

A STOCHASTIC MARKOV MODEL FOR PREDICTING CASH RECOVERIES ON A DEFAULTED RETAIL BANK PORTFOLIO

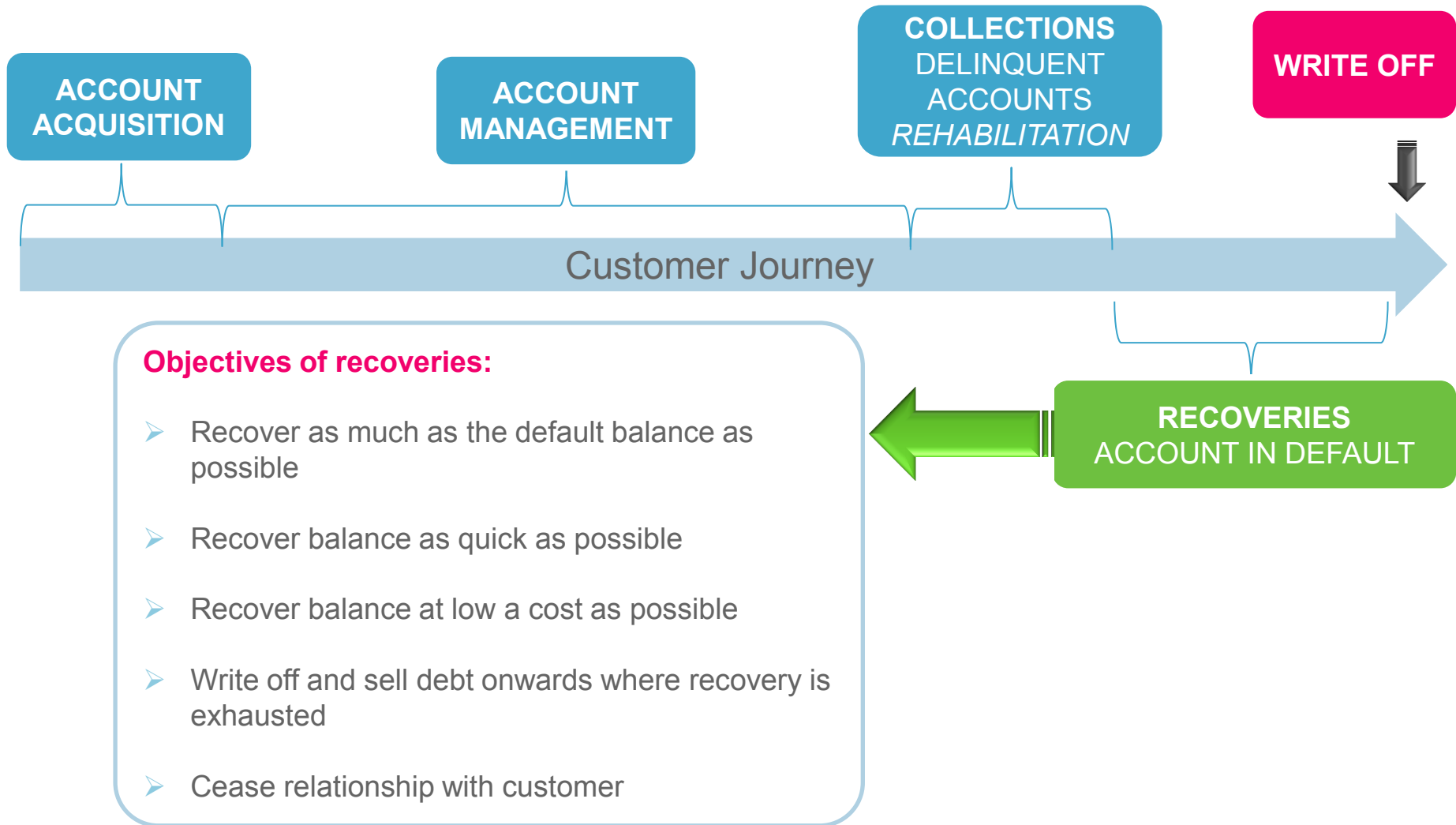
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Motivation

The Account Lifecycle

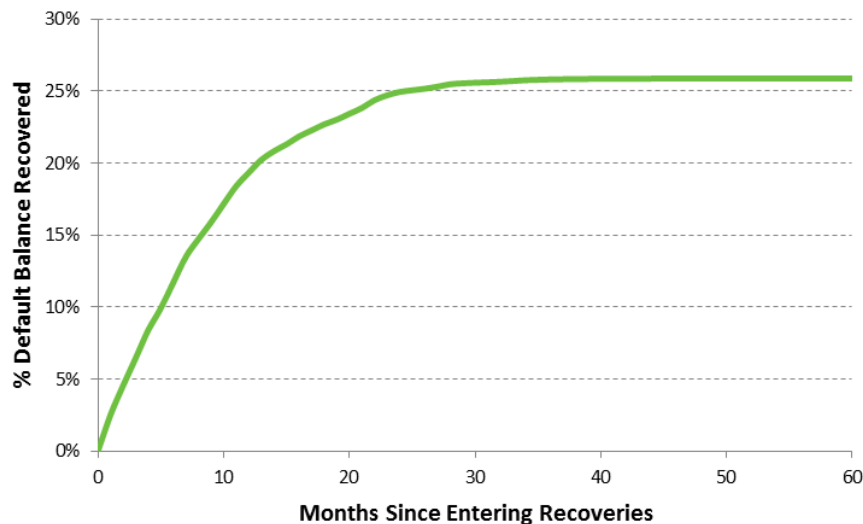


Motivation

Cash Recovery Curves

One minus the cash recovered divided by the default balance should be analogous to both:

- The raw (undiscounted) portfolio Loss Given Default
- The level of provision that should be held for accounts in the recoveries function



What this curve tells us:

- Recovery rate of **25.9%** of default balance
- Therefore, loss rate of **74.1%**
- Assuming monthly compounding and discounting cash flows by 12% per annum, we can obtain a discounted LGD of **76.3%**

Objective

We look to build a model that predicts, at a **portfolio** or **segment** level, the amount of exposure that has entered the recoveries function, that will be recovered. Moreover, we look to use portfolio behaviour in the first few months to segment the portfolio into behavioural states and predict the full recovery curve using a Markov model. This will be extended to a stochastic model with time dependency to incorporate randomness and ease of incorporating into a stress testing framework.

Motivation

Business Rationale

Traditionally, cash recovery curves have been constructed using a combination of segmentation and cohort analysis, and aims to find the “best estimated” recovery curve. The proposed Markov model looks to create a prediction for the recovery curve taking into account some key factors:

- Recovery is inter-related to the write-off rate and they need to be modelled together coherently. The write-off policy directly influences the amount of cash available to collect. I.e. once a balance has been written-off, it is not available to collect upon (not taking into account fortuitous recoveries)
- Recovery not only depends upon portfolio, but the behaviour of accounts within that particular portfolio
- Marginal recovery rates diminish over time and are volatile and uncertain
- Recovery rates are correlated with the economic climate
- Model can be help:
 - Drive write-off and recovery strategies
 - Optimise debt sale decisions
 - Calculate full and final balances for customers

Model Assumptions

- The recovery process is Markovian – i.e. the future recovery rates only depends on an accounts current status and not the path it has taken to get there
- Write-Off and Recovery are **absorbing** states – once a proportion of the balance of an account has entered either, it cannot move out of these states
- Excludes fortuitous recoveries

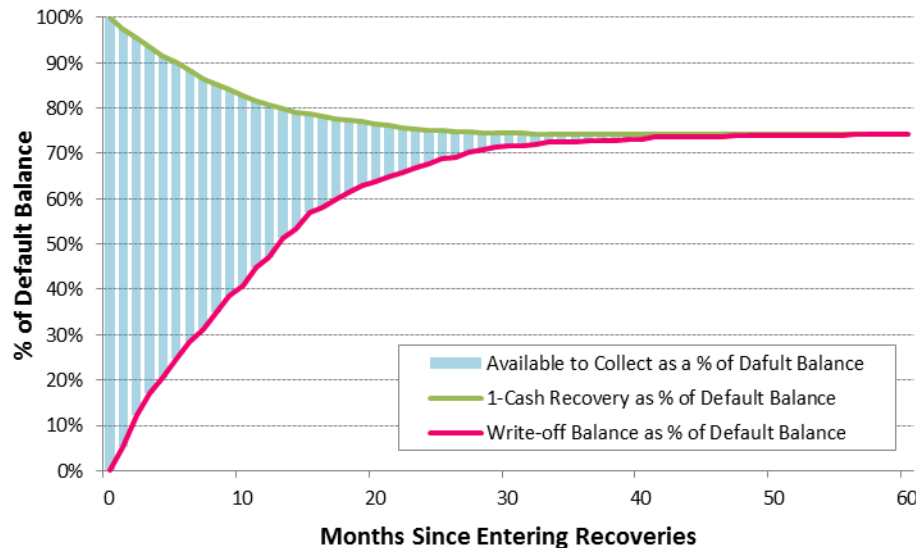


Model Structure

Extension to Cash Recovery Curves

As discussed, a dynamic that aids in the development of a cash recovery model is the amount of the default balance that has been written-off.

Lets now look at the cash recovered curve as $1 - \text{Cash Recovered as \% age of default balance}$ and add the write-off as $\%$ of default balance curve.



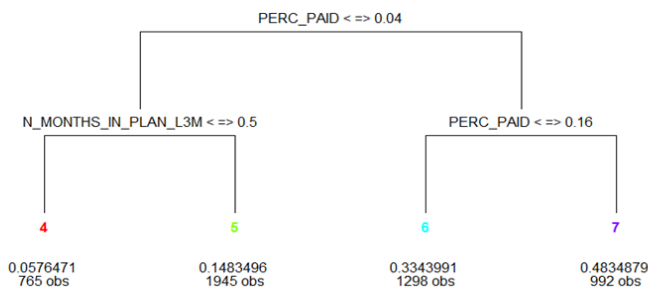
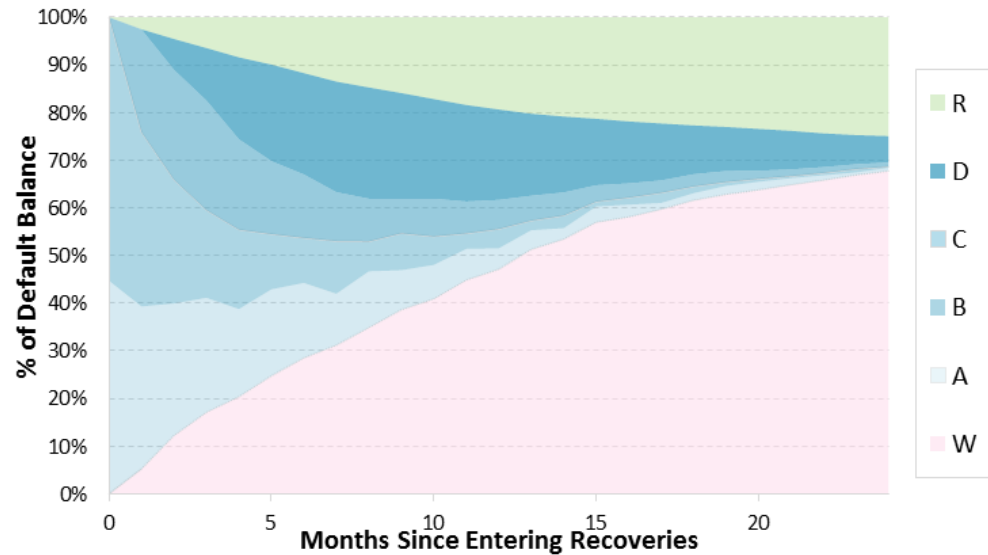
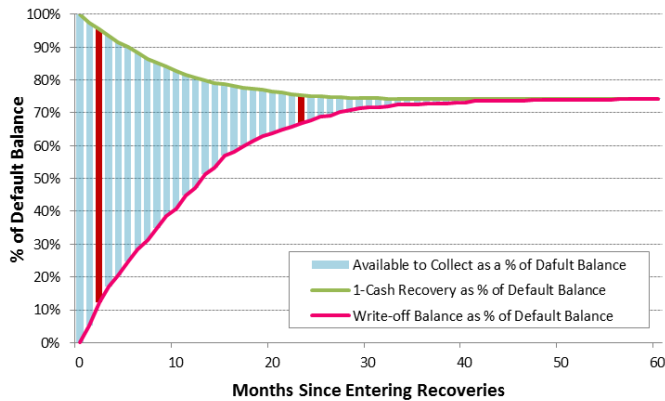
What this curve tells us:

- Same information as before can be extracted
- In addition, at any time, we can tell how much of the default balance is **available to collect** at any point in time
- For example, at month 12, we have recovered **19.3%** and written-off **47.1%**. So we have available **33.6%** of default balance to recover. So maximum possible recovery is **52.9%**

Building the Model

Behavioural “Likeliness to Pay” States

The Markov model requires that a number of “states” are defined before an estimate of the transition rates between these states are estimated. It is proposed that the states are defined by decision tree analysis based on account behaviour up to an arbitrary point with an outcome variable of cash recovered at a future arbitrary point. Recovery and Write-Off are absorbing states.



- Likeliness to Pay ↑
- R. Recovery
 - D. PERC_PAID >= 0.16
 - C. PERC_PAID in (0.04, 0.016]
 - B. PERC_PAID < 0.04 and N_MONTHS_IN_PLAN_L3M > 0
 - A. PERC_PAID < 0.04 and N_MONTHS_IN_PLAN_L3M = 0
 - W. Write-Off



Building the Model

Estimating the Transition Matrix

Transition Period

The transition matrix will look at **balance transitions** over a **6 month** period. 6 months has been chosen as it produces a more granular cash recovery curve if a longer period had been chosen, but reduces the volatility of the estimates if a shorter transition period is chosen.

Estimate Window

The transition rates have been estimate **between 6 months and 12 months** in recoveries.

This is has been chosen for 3 reasons:

- The model can be **applied after only 6 months of behaviour**
- Some of the “higher payer states” have **still to be hit** at 6 months.
- We will see later that the matrix is scaled to reflect diminishing likeliness to pay across all states as time progresses – choosing an earlier point to estimate the matrix makes allows the scaling factor to be fit to a “nice” **monotonic functional form**.

Transition Estimators

The 6 month balance transition rate T_{ij} can be estimated as follows:

$$\widehat{T}_{ij} = \frac{B_{ij}}{B_i}$$

Where B_{ij} is the total balance transitioned between state i at 6 months in recoveries to state j at 12 months in recoveries. B_i is the total balance in state i at 6 months in recoveries.



The Estimated Transition Matrix

The 6 month transition matrix **T** is estimate from our data below and shows that, as expected, the proportion of initial state balance recovered increases and the write-off rate decreases as the “quality” of the state increases.

A Few Observations

- An individual account can be in 2 states at once, Recovery and any other state
- Once a proportion of the balance has been paid, it cannot be withdrawn, similarly, once the balance has been written-off, it is not reversed.
- All or none of the balance is written-off, so the transition rate to write-off is much higher than to Recovery, as a proportion of the balance can transition.

Likliness to Pay ↑

R. Recovery
D. PERC_PAID >= 0.16
C. PERC_PAID in (0.04,0.016]
B. PERC_PAID < 0.04 and N_MONTHS_IN_PLAN_L3M > 0
A. PERC_PAID < 0.04 and N_MONTHS_IN_PLAN_L3M = 0
W. Write-Off

	A	B	C	D	W	R
A	15.6%	13.6%	14.7%	6.2%	44.5%	5.4%
B	20.8%	20.0%	7.0%	15.1%	27.9%	9.1%
C	0.0%	0.0%	23.6%	34.6%	26.9%	14.9%
D	0.0%	0.0%	0.0%	56.3%	25.3%	18.4%
W	0.0%	0.0%	0.0%	0.0%	100.0%	0.0%
R	0.0%	0.0%	0.0%	0.0%	0.0%	100.0%



Validating the Fit

The next step is to check that the fit of the model is adequate, or in other words, check that the Markov property holds.

For a general Markov Chain model, a full verification of the Markov property would involve a great deal of work and a lot of data. In practice, it is generally considered sufficient to look at triplets of successive observations.

Denote b_{ijk} as the proportion of balance that starts in state i at $t = a$ and transitions through state j at $t = a + 1$ and ending in state k at time $t = a + 2$, for all times a where $1 \leq a \leq N-2$ and the interval is 6 months. If the Markov property holds, we expect b_{ijk} to be an observation from a Binomial(b_{ij}, t_{jk}) distribution. The test is therefore a chi-squared goodness-of-fit test based on the test statistic:

$$\chi^2 = \sum_i \sum_j \sum_k \frac{(b_{ijk} - b_{ij}\hat{t}_{jk})^2}{b_{ij}\hat{t}_{jk}}$$

The number of degrees of freedom is given by $r-q+s-1$ where:

- s = number of states
- q = number of pairs (i,j) for which $b_{ij} > 0$
- r = number of triplets (i,j,k) for which $b_{ij}b_{jk} > 0$

For our example, it can be shown that the χ^2 statistic is **4.5** on **17** df, therefore there is no evidence, even at the 1% significance level, that model is not a reasonable fit.



Applying the Time Homogenous Model

Let $B(t)$ be the vector of balances at time t , then $B(n)$, the vector of balances at time n can be estimated as follows:

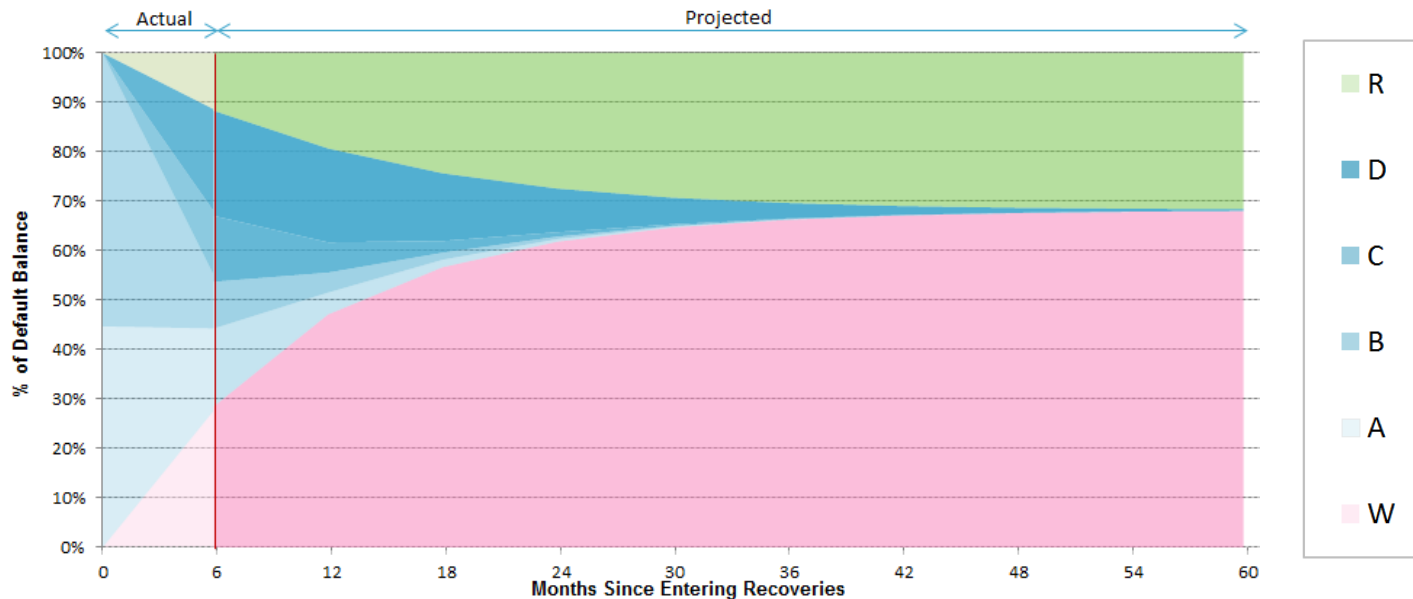
$$\widehat{B(n)} = B(t)T^{n-t}$$

Applying recursively from month 6 where $t = 1,2,3\dots$ that corresponds to months 6,12,18..., we get a sequence that gives us estimated balances at 6 month time steps:

$$\widehat{B(2)} = B(1)T$$

$$\widehat{B(3)} = B(2)T$$

Continuing the sequence up to time 10 (month 60) we obtain the following estimate of the recovery curve, write-off curve and % of balance in each state at each time t :



Issues with the Time Homogenous Model

0 to 6 Month Transition

	W	R
A	38.5%	7.4%
B	20.4%	15.2%
C	0.0%	0.0%
D	0.0%	0.0%

6 to 12 Month Transition

	W	R
A	44.5%	5.4%
B	27.9%	9.1%
C	26.9%	14.9%
D	25.3%	18.4%

12 to 18 Month Transition

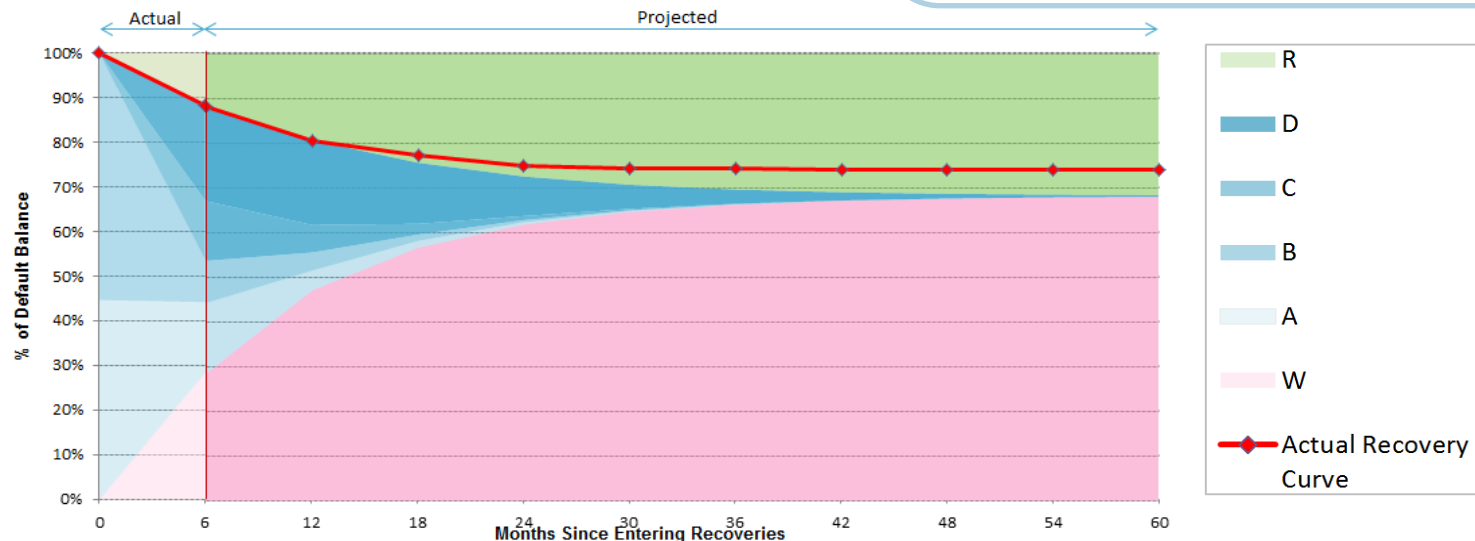
	W	R
A	41.7%	2.9%
B	36.7%	7.9%
C	52.4%	5.7%
D	41.9%	13.5%

18 to 24 Month Transition

	W	R
A	84.2%	0.9%
B	31.3%	2.5%
C	43.9%	7.9%
D	32.3%	19.4%

So what are the Issues?

- The model assumes that transition rates are constant and do not depend in time.
- However, we can see that in general, the 6m recovery rates for each state **diminishes** over time and write-offs rates increase with time.
- This leads to an **over-prediction** in the amount of cash recovered



A Time-Dependant Model

Let our transition matrix be T as before and let E be the matrix of left eigenvectors and D be the diagonal matrix of left eigenvalues of T , then there exists a constant μ such that when $\mu = 1$:

$$T = E (D^\mu) E^{-1}$$

When $\mu < 1$, the transition matrix T becomes a “lighter” recovery matrix. Moreover, a 50% decrease in μ roughly results in a 50% decrease in the recovery transitions. The aim is to find μ as a function of time such that the predicted cash recovery through the transition matrix is calibrated the actual cash recovery curve.

Calibrating μ to data

Define μ as a deterministic differential equation such that the change in μ satisfies the following equation:

$$\Delta\mu_t = a(b - \mu_t)\Delta t$$

Where b is the **long run equilibrium value of μ** and a is the **speed that the equation converges to b** . Then a and b can be found by linear programming where the objective function is defined as:

$$\sum_{i=j}^N (\text{Actual } CR_i - \text{Predicted } CR_i)^2$$

Where N the length of the recovery period is (60 in our example) and j is the end of the period that the original transition matrix has been calculated from (12 in our example). Subject to the following constraints:

- Write off rates and cash recovery rates are strictly increasing
- μ is a decreasing function of time

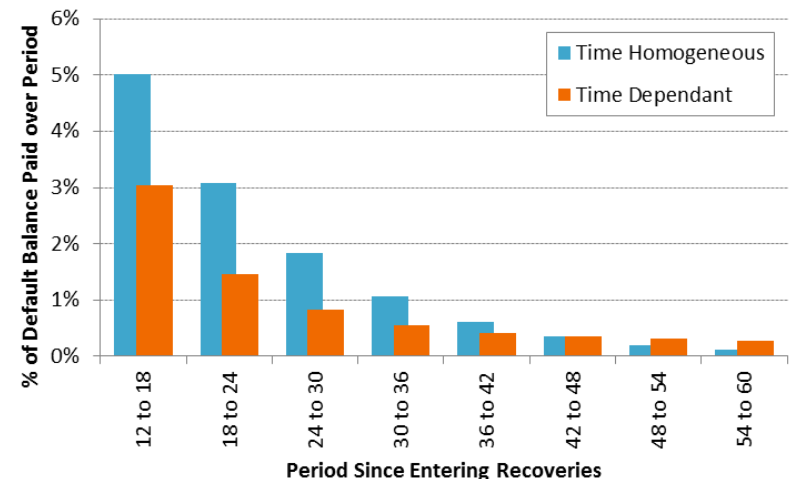
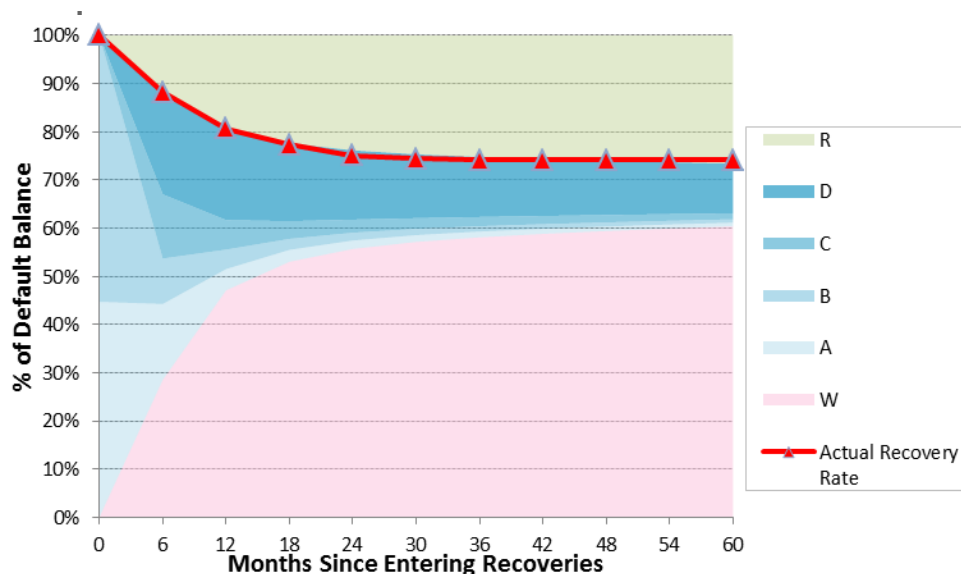
Our Example Continued

The transition above was calibrated to the actual cash recovery profile also shown above (red line in chart) and yielded the following equation for the μ parameter ($\Delta t = 1$):

$$\Delta\mu_t = 0.5(0.095 - \mu_t)$$

A Few Observations

- Cash Recovery has now been accurately predicted on the build sample
- But at the expense of predicting Write-Off Rates – issue still to be addressed
- Payment rates are reduced towards zero quicker than the time-homogeneous model (as expected!) and tend towards 0.095%.



A Stochastic Extension

We now look to capture some randomness in the cash recovery curve and produce a distribution of recovery rates. We extend to a discrete-time Cox-Ingersall-Ross process:

$$\Delta\mu_t = a(b - \mu_t)\Delta t + \sigma\sqrt{\mu_t}dw$$

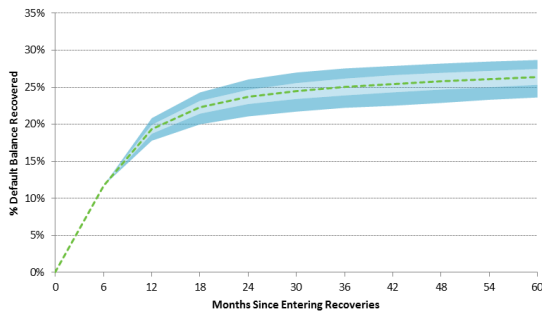
Where σ is the volatility of μ and dw is Wiener process. The larger the size of σ , the wider the distribution of the cash recovery curve.

μ will drift towards b as before, but with an element of randomness added.

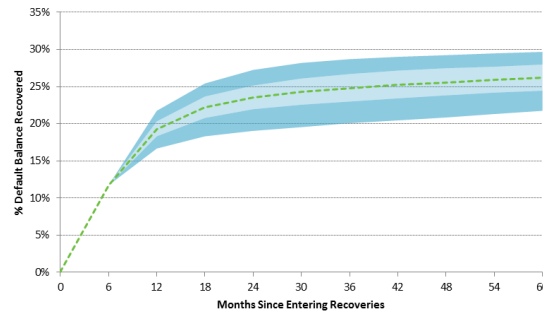
Why use this Equation?

- The It builds on the time-dependent case explored before
- The square root term always ensures that μ is positive
- The volatility of the change in μ is scaled by the value of μ , and so as μ approaches the long-term value, the volatility of the change in μ decreases. I.e. the marginal payment rate doesn't jump up at later points in the projection, on average.

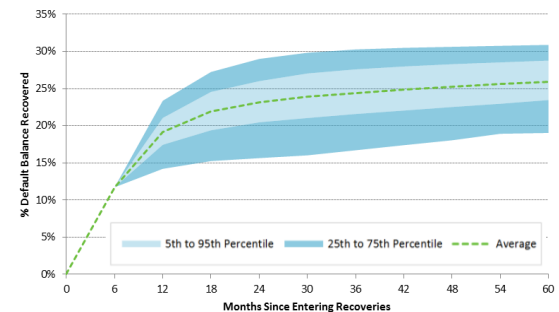
$\sigma = 15\%$



$\sigma = 25\%$



$\sigma = 45\%$



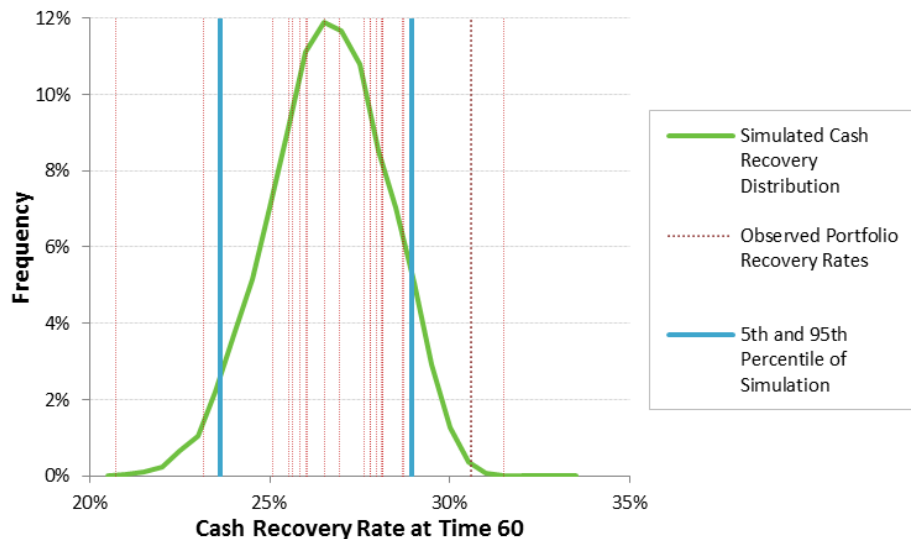
Calibrating the Volatility

Portfolios or cohorts of data with similar characteristics as the sample used in the model build sample should be used to calibrate the volatility parameter. In our example, we have looked 20 similar cohorts / portfolios with observed cash recovery outcomes.

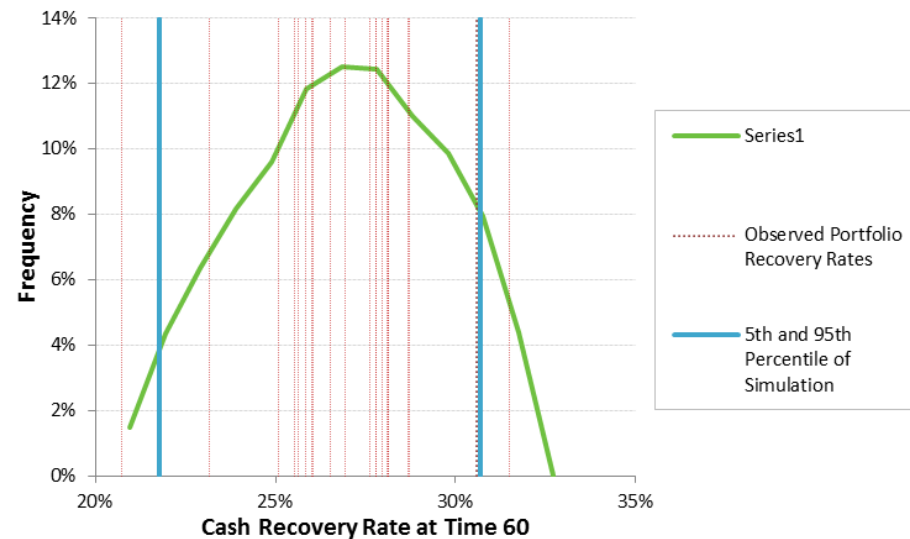
We therefore, must “pick” a volatility such that only 1 (5% of 20) of these observed portfolio recovery rates land in the simulated 5th percentile of the distribution and 1 land in the 95th percentile of the distribution.

We 10,000 trials use trial and error to find this volatility. We can see that using 20% volatility gives 2 observations in the lower tail and 2 in the upper tail and therefore needed to be increased. We can see that using 45% volatility meets out requirements.

$\sigma = 20\%$



$\sigma = 45\%$



Stress Testing Recovery Rates

For the purpose of this example, we look at the change in GDP as the scenario “driver”. We first must find the “baseline” scenario, and look at a stress scenario relative to this.

The baseline scenario should be defined as the 6-month GDP growth rate over the model build sample timeframe. We must also assume a GDP volatility. Here we have used **2.2%**, the historical GDP growth rate volatility.

Finally we can define our scenarios. Our example looks at two separate scenarios:

	12	18	24	30	36	42	48	54	60
Model Build Sample GDP Growth	2.39%	1.30%	0.25%	1.25%	1.00%	0.20%	0.35%	1.50%	1.00%
2008 Recession Scenario	-2.67%	-3.64%	0.30%	0.99%	1.28%	0.10%	0.68%	-0.19%	0.58%
GDP Growth Scenario	2.50%	2.50%	2.50%	2.50%	2.50%	2.50%	2.50%	2.50%	2.50%

We can now define our GDP shocks as a standard normal variant as follows:

$$\text{Shock}(t) = [\text{Scenario Growth Rate}(t) - \text{Baseline Growth Rate}(t)] / \text{GDP Volatility}$$

And gives us the following deterministic shocks:

	12	18	24	30	36	42	48	54	60
2008 Recession Scenario	- 2.281	- 2.228	0.023	- 0.115	0.125	- 0.046	0.149	- 0.764	- 0.189
GDP Growth Scenario	0.051	0.541	1.015	0.564	0.677	1.037	0.970	0.451	0.677



Stress Testing Recovery Rates

We then must define the correlation between the change in the 1 period payment rate (or μ) and the change in GDP. Taking **0.7** as an example gives us the correlation matrix R as follows:

$$R = \begin{bmatrix} 1 & 0.7 \\ 0.7 & 1 \end{bmatrix}$$

Define C as the Cholesky Matrix, such that:

$$C = LDL^T$$

Where L is the Lower Triangle Matrix of R, and define S as the vector of uncorrelated shocks (deterministic for GDP and random for μ), then a vector of **Correlated** shocks S^* is defined as:

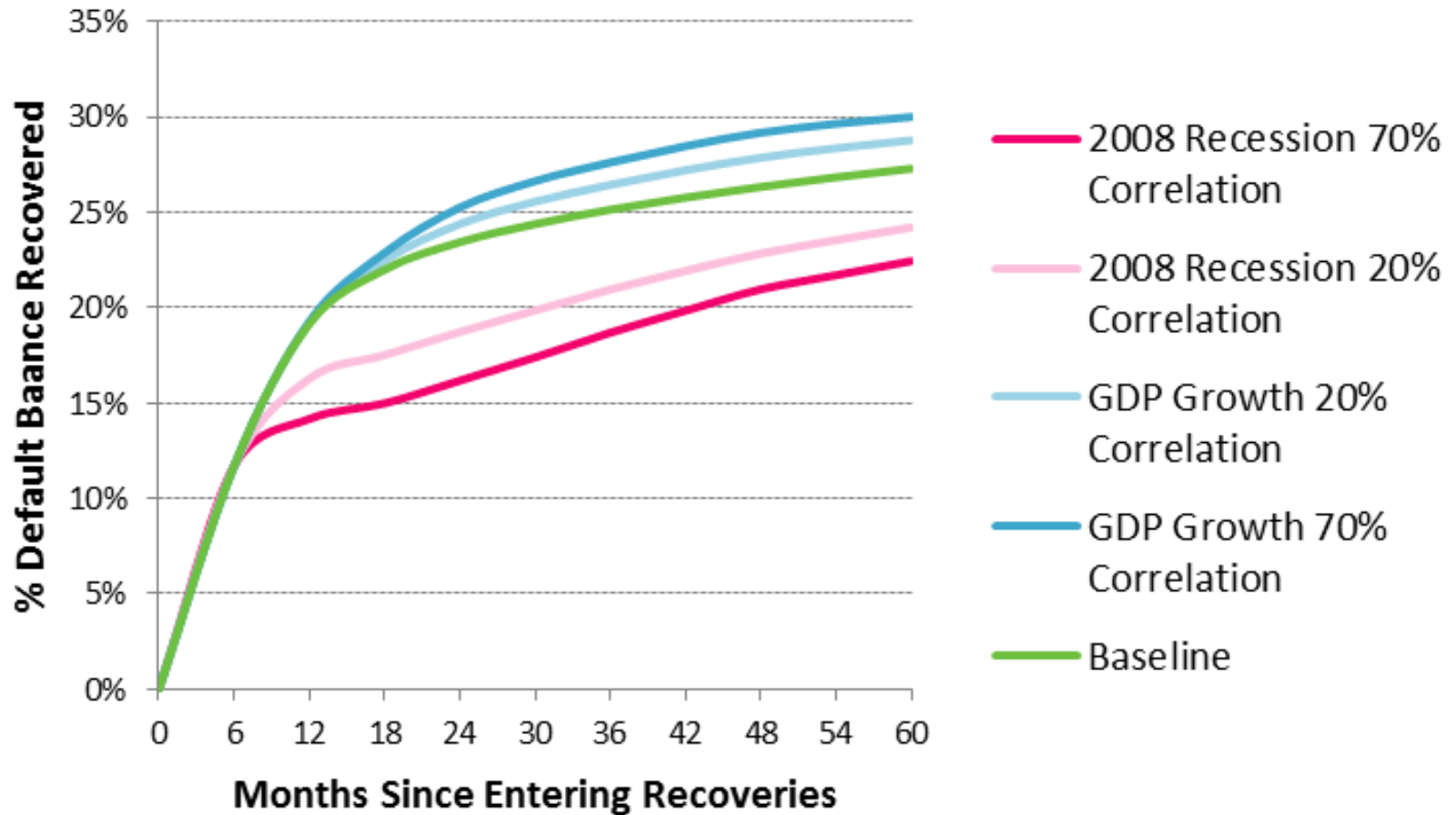
$$S^* = SC$$

We now rerun our stochastic projection for cash recovery as before, but using the random shock for μ as defined in S^* .

Here is the results of running the model (same parameters as before), but correlating it to the scenarios define before. As expected, the recovery rates fall when GDP decreases and vice-versa. We look at both a 20% and 70% correlation assumption.

Stress Testing Recovery Rates

Average Recovery Rate over 10,000 trials



Conclusions

Time – Homogeneous

- ✓ Simple to estimate
- ✓ Simple to explain
- ✓ Captures initial payment behaviour
- ✗ Doesn't capture diminishing payment rates across all behavioural states
- ✗ Often leads to over-estimation of recovery rate
- ✗ Requires adequate data volumes
- ✗ Model can only be applied to similar portfolios or cohort, so a number of models may need to be built on a number of segments.

Conclusions

Time – Dependant

- ✓ Builds on time homogeneous model
- ✓ Simple functional form
- ✓ Captures diminishing payment rates
- ✓ Better predicts cash recovery rates
- X More parameters to estimate and requires use of linear programming techniques
- X Write-off rate predictions sacrificed



Conclusions

Stochastic Model

- ✓ Builds on Time-dependant model
- ✓ Provides a distribution of recovery rates
- ✓ Can easily be extended into a stress testing framework
- X Requires a number of portfolios to estimate volatility
- X Correlations difficult to estimate
- X Difficult to implement and explain

