Calculating optimal limits for transacting credit card customers

Credit Scoring & Credit Control XIV
The University of Edinburgh

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School of Mathematics & Statistics
The University of Melbourne, Australia

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Outline

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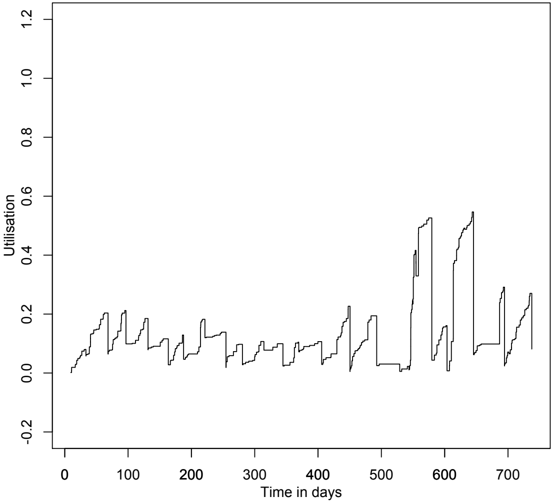
Transaction-level data can offer a more detailed picture of how a customer uses a credit card.

This talk presents a methodology for using transaction data to:

- Calculate an profit-maximising limit for existing transacting customers at an individual level, and;
- Gain insight into how a customer's purchasing behaviour affects their profitability.

Actual data shows a variety of behaviours



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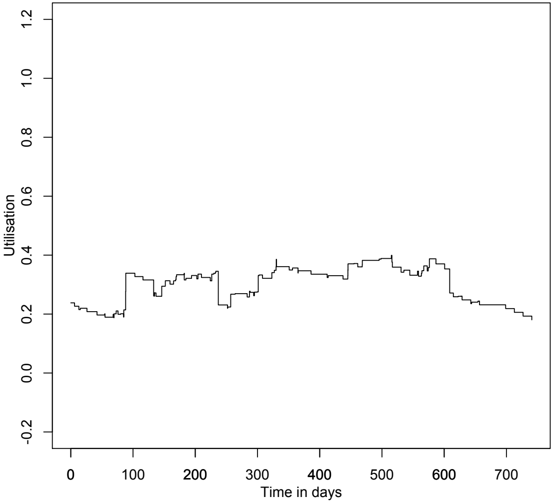

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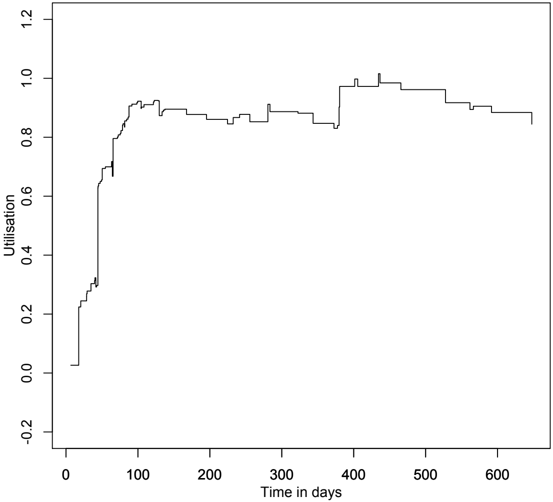
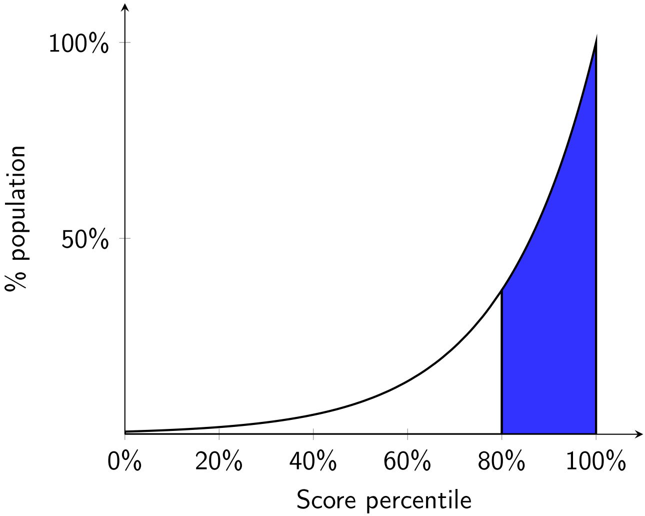


Strategies typically target top 20 — 30% of scores



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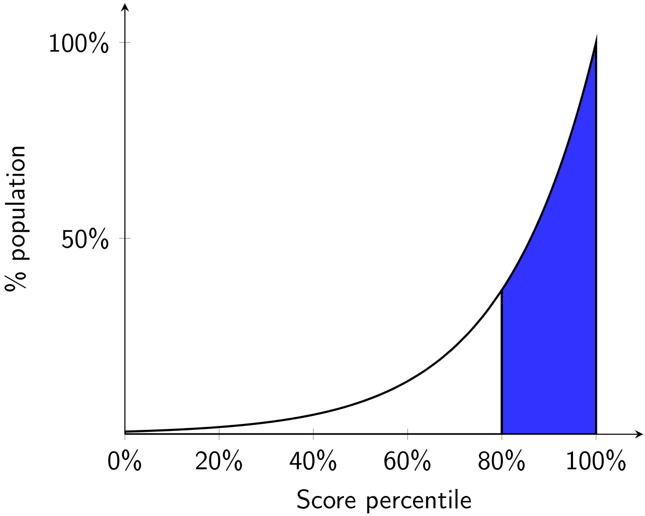

These are most likely to be transactors.

Transactor profit is driven by interchange and funding cost

The expected profit for an individual transactor in a single period of length T is

$$\mathbf{E}[R(T)] = \gamma\,\mathbf{E}[B_\ell(T)] - \nu\ell, \tag{1}$$

where

- γ is the interchange rate;
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- $B_\ell(T)$ is the balance of the credit card at time T with credit limit ℓ.

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We want to find the limit which maximises Equation (1),

$$\hat{\ell} := \arg\max_{\ell \in \Lambda} \left\{ \gamma\,\mathbf{E}[B_\ell(T)] - \nu\ell \right\},$$

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How can we calculate $\mathbf{E}[B_\ell(T)]$?

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Close, but $\mathbf{E}[A(T) \wedge \ell] \neq \mathbf{E}[B_\ell(T)]$.

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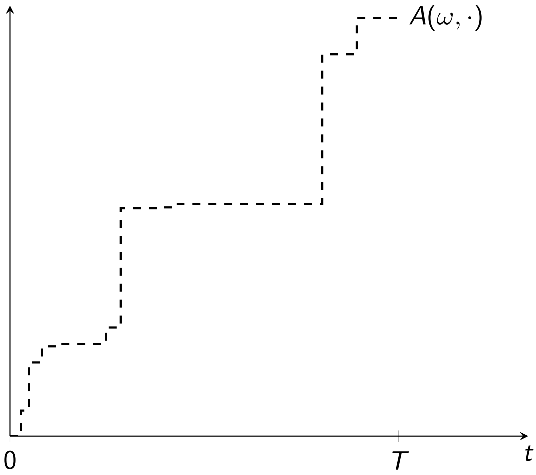
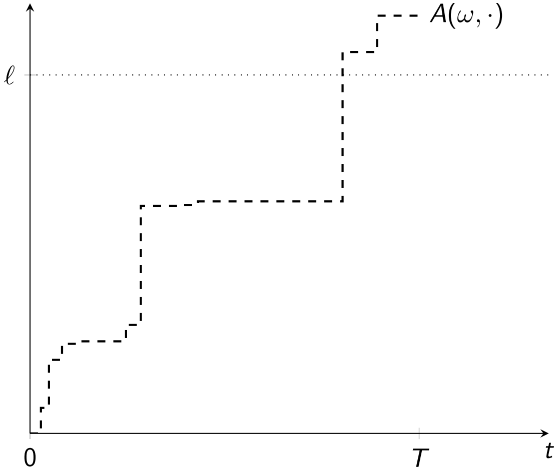
The real process is bounded by the newsvendor model and our model.

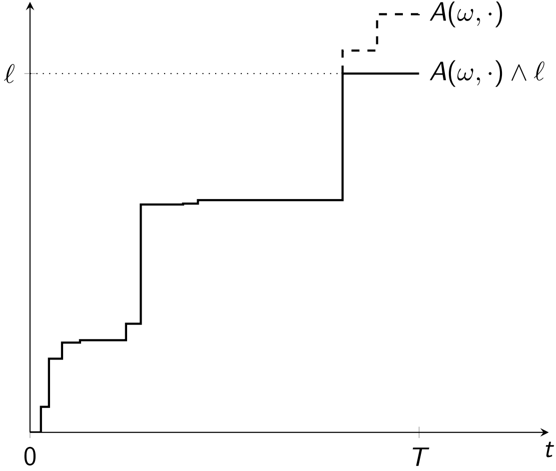
Attempted purchases $A(T)$



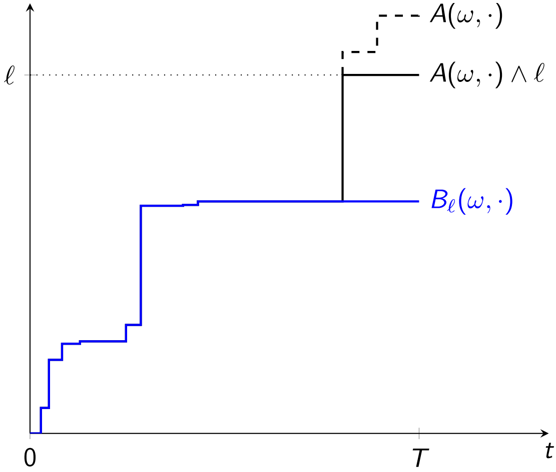
Purchases above the limit are declined



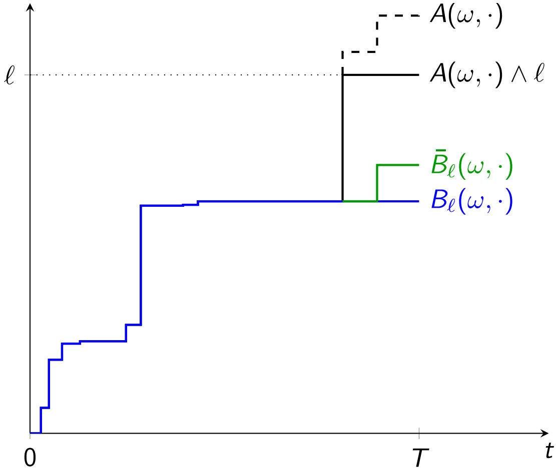
Newsvendor model takes the minimum of the demand and ℓ



$B_\ell(t)$ freezes the process at its last level



True process lies between our model and the newsvendor



Some details on the calculation of the distribution of $B_\ell(t)$ and its expectation are in the appendix.

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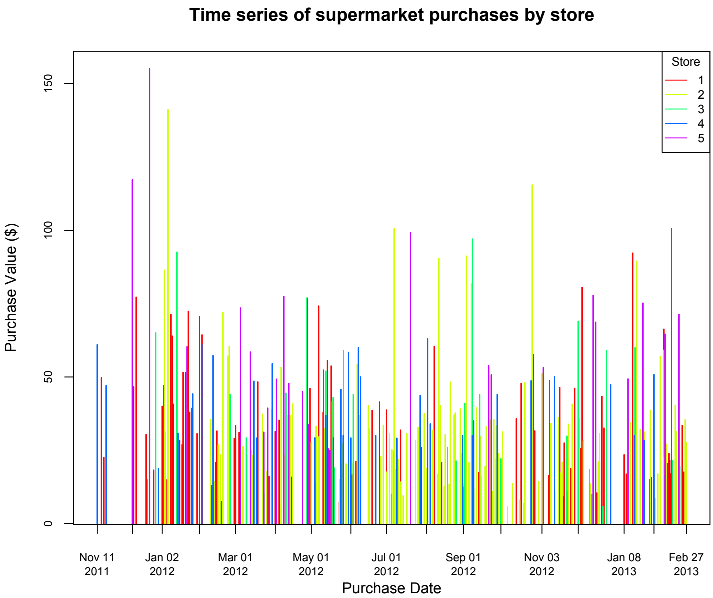
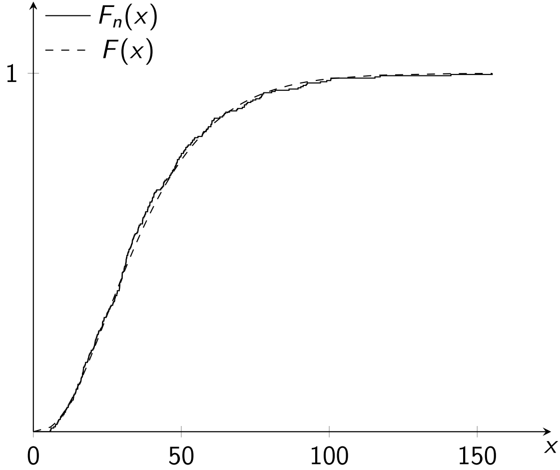
To calculate the optimal limit, we use their transaction data to understand the frequency and value of their purchases.

Supermarket time series



Γ-distribution a good fit to supermarket purchases

Parameters $\hat{k} = 0.0769$ (shape) and $\hat{\mu} = 2.8946$ (scale).



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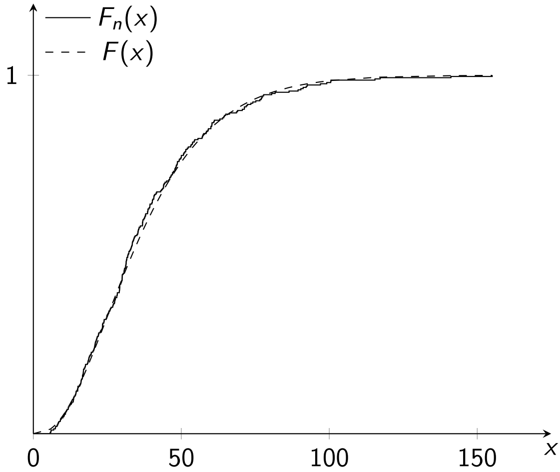

We assumed purchase times followed a Poisson process with estimated rate $\hat{\lambda} = 0.6451$.

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The results assume no change in the customer's purchasing behaviour.

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2. Ask customers with low utilisation to opt in to a lower limit. Reserve their original limit for 12 – 24 months and allow them to reinstate it upon request.

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More detail in our paper

Budd, J. K. & Taylor, P. G. "Calculating optimal limits for transacting credit card customers." arXiv preprint arXiv:1506.05376 (2015). http://arxiv.org/abs/1506.05376.

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- Modelling how purchasing behaviour changes following a limit change;
- Multiple and random payment times;
- Intensity-varying purchasing process.

Acknowledgments

Thanks to the School of Mathematics & Statistics at the University of Melbourne for providing funding to attend this conference.

P. G. Taylor's research is supported by the Australian Research Council (ARC) Laureate Fellowship FL130100039 and the ARC Centre of Excellence for Mathematical and Statistical Frontiers (ACEMS).

Appendix

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2. F and G are of exponential order and all moments of the distributions exist. These conditions are sufficient to ensure the existence of the Laplace transforms

$$\tilde{f}(\theta) = \int_0^\infty \mathrm{e}^{-\theta z}\, F(\mathrm{d}z) \quad \text{and} \quad \tilde{g}(\varphi) = \int_0^\infty \mathrm{e}^{-\varphi u}\, G(\mathrm{d}u).$$

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Integrating over τ and ξ gives us the following integral equation,

$$\begin{aligned} \mathrm{P}\big(B_\ell(t) \in (y,\ell]\big) &= \int_0^t \int_0^y \mathrm{P}\big(z + B_{\ell-z}(t-u) \in (y, \ell - z]\big)\, F(\mathrm{d}z)\, G(\mathrm{d}u) \\ &\quad + \int_0^t \int_y^\ell F(\mathrm{d}z)\, G(\mathrm{d}u). \end{aligned}$$

Solution to the integral equation

Define the three dimensional Laplace transform

$$\tilde{S}(\theta, \varphi, \psi) = \int_0^\infty \int_0^\infty \int_0^\ell \mathrm{e}^{-\varphi t - \theta \ell - \psi y} \, \mathsf{P}\big(B_\ell(t) \in (y, \ell]\big) \, \mathrm{d}y \, \mathrm{d}\ell \, \mathrm{d}t.$$

Solution to the integral equation

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$$\tilde{S}(\theta, \varphi, \psi) = \int_0^\infty \int_0^\infty \int_0^\ell \mathrm{e}^{-\varphi t - \theta \ell - \psi y} \, \mathsf{P}\big(B_\ell(t) \in (y, \ell]\big) \, \mathrm{d}y \, \mathrm{d}\ell \, \mathrm{d}t.$$

Apply the transform to the integral equation to obtain

$$\tilde{S}(\theta, \varphi, \psi) = \frac{1}{\theta \varphi \psi} \frac{\tilde{g}(\varphi)\big(\tilde{f}(\theta) - \tilde{f}(\theta + \psi)\big)}{1 - \tilde{g}(\varphi)\tilde{f}(\theta + \psi)}.$$

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We can invert the last expression numerically in order to solve our optimisation problem. Multiplying the transform of the expectation by θ yields the transform of its derivative.