Inflated mixture models: Applications to multimodality in loss given default.

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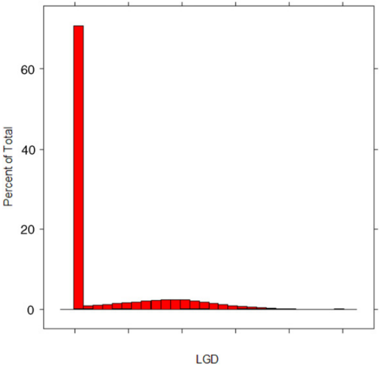
Holyrood Park

It may give us some idea about what is modality...

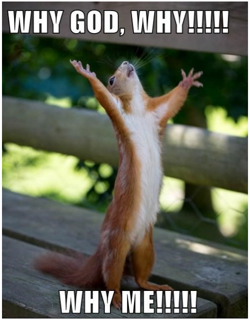
Agenda

Illustrative examples of LGD distributions I

- Tong, Edward NC, Christophe Mues, and Lyn Thomas. "A zero-adjusted gamma model for mortgage loan loss given default."International Journal of Forecasting 29.4 (2013): 548-562.



Why zero-adjusted?



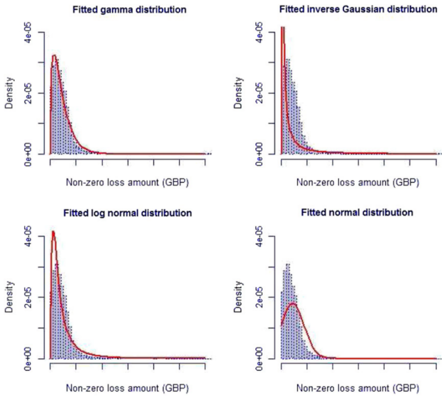
- Because the support of the gamma distribution does not include the zero point, i.e., the support is $(0, +\infty)$.

How zero-adjusted?

$$f_{i_{gamma}}(y;\vartheta) = \begin{cases} \delta_0, & \text{if} \quad y = 0 \\ (1-\delta_0) f_{gamma}, & \text{if} \quad 0 < y < +\infty, \end{cases}$$

where f_{gamma} follows a gamma distributions.

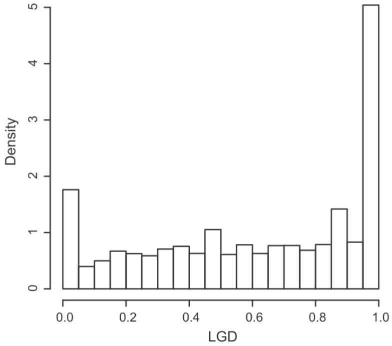
We've got more?



- Inverse Gaussian distribution, with support $(0, +\infty)$.
- Log normal distribution, with support $(0, +\infty)$.
- (truncated) Normal Gaussian distribution, with support $[0, +\infty)$.

Illustrative examples of LGD distributions II

- Calabrese, Raffaella. "Downturn loss given default: Mixture distribution estimation."European Journal of Operational Research 237.1 (2014): 271-277.



Agenda

Who is beta? and Mixture? and Inflated Mixture?

The beta distribution

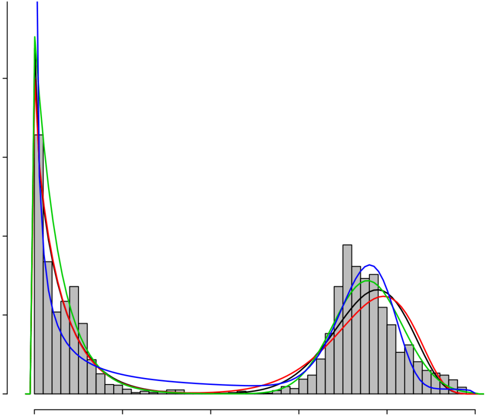
The well-known beta distribution has 2 parameters, mean $\mu \in (0, 1)$, and precision $\phi > 0$. Its density function is defined for $y \in (0, 1)$, where $\Gamma(.)$ is the gamma function:

$$f(y; \mu, \phi) = \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1-\mu)\phi)} y^{(\mu\phi-1)}(1-y)^{(1-\mu)\phi-1}.$$

The mixture of beta distributions

Given $f_1(y\,;\mu_1,\phi_1)$ and $f_2(y\,;\mu_2,\phi_2)$, two beta distributions, we set a mixture of two beta distributions, now with 5 parameters!:

$$f_{m_2b}(y\,;\pi,\mu_1,\phi_1,\mu_2,\phi_2)=\pi f_1(y\,;\mu_1,\phi_1)+(1-\pi)f_2(y\,;\mu_2,\phi_2).$$



The inflated mixture of beta distributions

The Y distribution is said to be an inflated (in zeros and ones) mixture of two beta distributions, with a 7-parameter $\vartheta = (\delta_0, \delta_1, \pi, \mu_1, \phi_1, \mu_2, \phi_2)$, if its density function is given by:

$$f_{im_2b}(y;\vartheta) = \begin{cases} \delta_0, & \text{if} \quad y = 0 \\ (1-\delta_0-\delta_1) f_{m_2b}, & \text{if} \quad 0 < y < 1 \\ \delta_1, & \text{if} \quad y = 1, \end{cases}$$

where f_{m_2b} follows a mixture of two beta distributions.

The first moment (mean) of Y, given by

$$E[Y] = (1-\delta_0-\delta_1)(\pi\mu_1 + (1-\pi)\mu_2) + \delta_1.$$

Parameter estimation

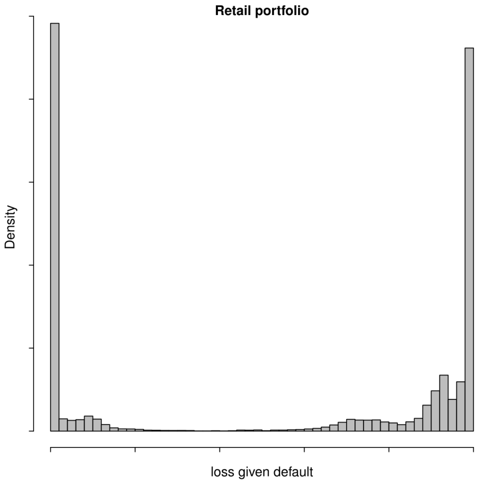
Parameter estimation is performed by straightforward use of maximum likelihood estimation (MLE).

Despite the existence of competitive methods, see for example expectation-maximization (EM) algorithm, our simulation studies support simple application of MLE approach.

The likelihood function of the inflated mixture model f_{im_2b}, with a vector 7-parameter $\vartheta = (\delta_0, \delta_1, \pi, \mu_1, \phi_1, \mu_2, \phi_2)$, is based on a sample of n observations, $\mathcal{D} = \{y_i\}$, independent and identically distributed:

$$L(\vartheta; \mathcal{D}) \propto \prod f_{im_2b}(y_i; \delta_0, \delta_1, \pi, \mu_1, \phi_1, \mu_2, \phi_2).$$

Brazilian bank non-performing retail data

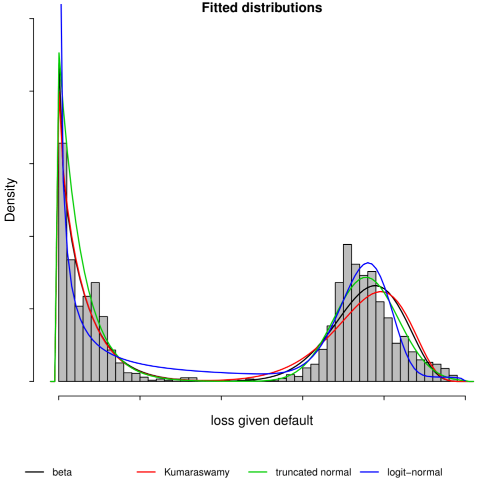


Brazilian bank non-performing retail data

Tabela: Summary of observed LGD data.

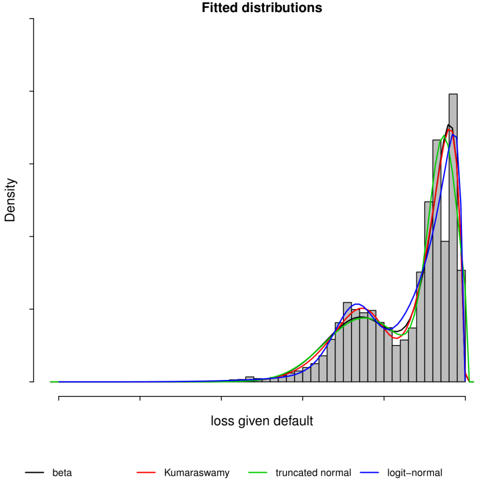
Portfolio	Qtd	Mean	Median	SD	#0	#1
1	15,295	0.52195	0.7272	0.4746	5722	6634
2	22,951	0.59814	0.9093	0.4596	8349	8398
3	440	0.32945	0.7466	0.4004	232	44
4	2,991	0.72060	0.9175	0.3810	510	265

Portfolio 1



Fitted distributions
Density
loss given default
beta
Kumaraswamy
truncated normal
logit−normal

Portfolio 2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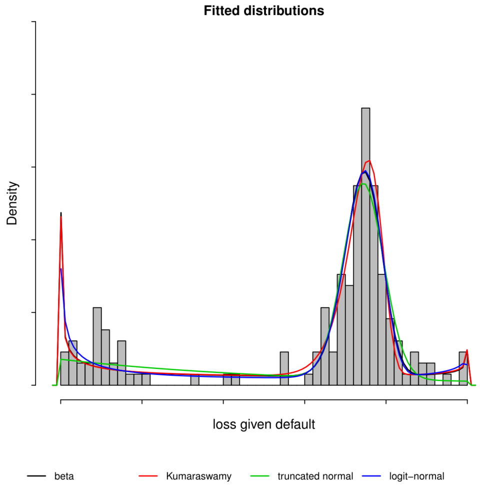
Portfolio 3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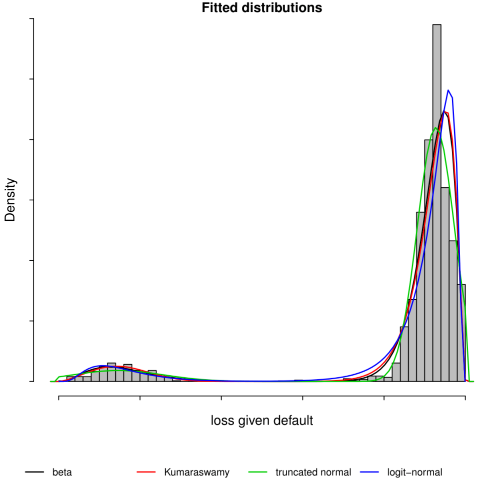
Portfolio 4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Brazilian bank non-performing retail data

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Tabela: Expected mean LGD by portfolio and by model.

Portfolio	Expected lgd			
	beta	Kumaraswamy	Truncated normal	Logit-normal
1	0.52205	0.52173	0.52187	0.52272
2	0.59810	0.59817	0.59814	0.59789
3	0.33187	0.33129	0.32993	0.33088
4	0.72072	0.72021	0.72060	0.71841
Relative difference %	**0.1912%**	0.1167%	0.0330%	0.0590%

Agenda

Our contribution: The regression model version

$$f_{im_2b}(y;\vartheta) = \begin{cases} \delta_0, & \text{if} \quad y = 0 \\ (1-\delta_0-\delta_1) f_{m_2b}, & \text{if} \quad 0 < y < 1 \\ \delta_1, & \text{if} \quad y = 1, \end{cases}$$

$$\begin{cases} (\delta_{0i}, \delta_{1i}) & = & \left(\frac{e^{\mathbf{x}_{1i}^\top \beta_1}}{1+e^{\mathbf{x}_{1i}^\top \beta_1}+e^{\mathbf{x}_{2i}^\top \beta_2}}, \frac{e^{\mathbf{x}_{2i}^\top \beta_2}}{1+e^{\mathbf{x}_{1i}^\top \beta_1}+e^{\mathbf{x}_{2i}^\top \beta_2}} \right) \\ \mu_{1i} & = & \frac{e^{\mathbf{x}_{3i}^\top \beta_3}}{1+e^{\mathbf{x}_{3i}^\top \beta_3}} \\ \mu_{2i} & = & \frac{e^{\mathbf{x}_{4i}^\top \beta_4}}{1+e^{\mathbf{x}_{4i}^\top \beta_4}}, \end{cases}$$

Note that the inflated mixture of beta regression model can be viewed as an extension of the inflated beta regression model introduced by Ospina, R. & Ferrari, S. L. (2012). A general class of zero-or-one inflated beta regression models. Computational Statistics & Data Analysis, 56(6), 1609-1623.

Application data

- We consider a sample of the portfolio 1, containing 5000 retail loans.
- We consider two covariates made available by the bank.
- Let $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$, where $\mathbf{x_1}$ is the interceptor parameter, i.e., $\mathbf{x_1} = 1$, and two others are real covariates.
- $\mathbf{x_2}$ represents two group of clients according to the behavioural risk presented. The bank has its behaviour score model and segregated their customers into two groups, roughly, $\mathbf{x_2} = 0$ to customers with poor credit risk and $\mathbf{x_2} = 1$ with better credit risk.
- $\mathbf{x_3}$ is related to the loan characteristics. The loan classified as $\mathbf{x_3} = 0$, represents a group of loans with term relatively shorter than the group with $\mathbf{x_3} = 1$.

Application data

Thus, we use the following setting of the link functions:

$$
\begin{cases}
\delta_{0i} & = & \dfrac{e^{(\beta_{11}\mathbf{x}_{1i}+\beta_{12}\mathbf{x}_{2i}+\beta_{13}\mathbf{x}_{3i})}}{1+e^{(\beta_{11}\mathbf{x}_{1i}+\beta_{12}\mathbf{x}_{2i}+\beta_{13}\mathbf{x}_{3i})}+e^{(\beta_{21}\mathbf{x}_{1i}+\beta_{22}\mathbf{x}_{2i}+\beta_{23}\mathbf{x}_{3i})}} \\
\delta_{1i} & = & \dfrac{e^{(\beta_{21}\mathbf{x}_{1i}+\beta_{22}\mathbf{x}_{2i}+\beta_{23}\mathbf{x}_{3i})}}{1+e^{(\beta_{11}\mathbf{x}_{1i}+\beta_{12}\mathbf{x}_{2i}+\beta_{13}\mathbf{x}_{3i})}+e^{(\beta_{21}\mathbf{x}_{1i}+\beta_{22}\mathbf{x}_{2i}+\beta_{23}\mathbf{x}_{3i})}} \\
\mu_{1i} & = & \dfrac{e^{(\beta_{31}\mathbf{x}_{1i}+\beta_{32}\mathbf{x}_{2i}+\beta_{33}\mathbf{x}_{3i})}}{1+e^{(\beta_{31}\mathbf{x}_{1i}+\beta_{32}\mathbf{x}_{2i}+\beta_{33}\mathbf{x}_{3i})}} \\
\mu_{2i} & = & \dfrac{e^{(\beta_{41}\mathbf{x}_{1i}+\beta_{42}\mathbf{x}_{2i}+\beta_{43}\mathbf{x}_{3i})}}{1+e^{(\beta_{41}\mathbf{x}_{1i}+\beta_{42}\mathbf{x}_{2i}+\beta_{43}\mathbf{x}_{3i})}}
\end{cases}
$$

The results

The results corroborate the finding that longer term loans held by lower credit risk clients have a much lower loss given default that the remaining group.

Tabela: Summary of average LGD estimated by the inflated mixture of two beta regression model

Portfolio 1	Subgroups		Qtd	Observed	Estimated
mean lgd = 0.5216	$\mathbf{x}_2 = 0$	$\mathbf{x}_3 = 0$	1,266	0.6325	0.6383
		$\mathbf{x}_3 = 1$	1,269	0.6324	0.6255
	$\mathbf{x}_2 = 1$	$\mathbf{x}_3 = 0$	1,234	0.4262	0.4265
		$\mathbf{x}_3 = 1$	1,231	0.3890	0.4101

The results: estimated parameters

Tabela: MLE results for the inflated mixture of two beta regression models

Parameter	Estimative (est)	Standard error (se)	$\|est\|/$ se
β_{11}	0.1890	0.0707	2.6731
β_{12}	0.8122	0.0809	10.0333
β_{13}	0.0851	0.0802	1.0599
β_{21}	0.9171	0.0641	14.3059
β_{22}	-0.2448	0.0786	3.1132
β_{23}	0.0013	0.0777	0.0173
π	0.5784	0.0828	6.9789
β_{31}	-0.7320	0.1005	7.2790
β_{32}	-0.2577	0.1086	2.3727
β_{33}	0.0728	0.1079	0.6750
ϕ_1	1.1757	0.0538	21.8377
β_{41}	1.1076	0.0293	37.7084
β_{41}	-0.1125	0.0349	3.2177
β_{41}	-0.0508	0.0351	1.4478
ϕ_2	62.8578	0.1110	566.0437

Selected Reference

- Calabrese, R.
  Downturn loss given default: Mixture distribution estimation.
  European Journal of Operational Research, 237(1), 271–277, 2014.
- Ospina, R. & Ferrari, S. L.
  A general class of zero-or-one inflated beta regression models.
  Computational Statistics & Data Analysis, 56(6), 1609–1623, 2012.
- Tong, Edward NC, Christophe Mues, and Lyn Thomas.
  A zero-adjusted gamma model for mortgage loan loss given default.
  International Journal of Forecasting, 29(4), 548–562, 2013.

Contact

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Oliveira, M.R.[1], Louzada, F.[2], Pereira, G.H.A.[3], Moreira, F.[4], Calabrese, R.[5],
Inflated Mixture Models: Applications to Multimodality in Loss Given Default (July, 2015). Available at SSRN: http://ssrn.com/abstract=2634919

Thank you!

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