The Art of PD Curve Calibration

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Credit Scoring and Credit Control XIII
Edinburgh
August 2013

[1] The opinions expressed in this presentation are those of the author and do not necessarily reflect views of the Bank of England.

Outline

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Data

S&P rating frequencies (%) and default rates (%) in 2009 and rating frequencies in 2010[2].

	2009		2010			2009		2010	
Rating	Freq	DR	Freq	DR	Rating	Freq	DR	Freq	DR
AAA	1.38	0	1.3	?	BBB-	7.83	1.09	7.79	?
AA+	0.63	0	0.45	?	BB+	4.54	0	4.6	?
AA	3.21	0	2.59	?	BB	5.03	1.02	5	?
AA-	4.18	0	3.78	?	BB-	7.53	0.91	6.86	?
A+	5.8	0.29	6.39	?	B+	7.47	5.48	7.12	?
A	8.7	0.39	8.58	?	B	8.23	9.96	7.9	?
A-	9.32	0	9.56	?	B-	5.17	17.16	5.25	?
BBB+	8.5	0.4	8.28	?	CCC-C	3.24	48.42	3.98	?
BBB	9.23	0.18	10.56	?	All	100	3.99	100	**1.14**

[2] Sources: [S&P(2010)], tables 51 to 53, and [S&P(2011)], tables 50 to 52

Problem

- **Forecast grade-level default rates** for 2010, based on
  - rating frequencies and grade-level default rates observed in 2009, and
  - rating frequencies observed at the beginning of 2010.
- Consider two cases:
  1. Overall default rate for 2010 is not known (hence to be forecast).
  2. An independent forecast of the overall default rate for 2010 is available.
- Economically motivated constraints for forecasts:
  - Forecast default rates may be small but must be positive.
  - Forecast default rates must strictly increase with lower creditworthiness.

Observations from slide 4

- Rating frequencies in 2009 and 2010 are statistically significantly different.
- The overall default rate cannot be considered constant.
- The empirical default rates can be zero.
- The empirical default rates need not be monotonous.
- **Consequence:** The default rates observed in 2009 are not likely to be good forecasts of the default rates in 2010.

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One-period model

- Pair of random variables (X, S):
  - Rating grade at beginning of period $X \in \{1, 2, \ldots, k\}$
  - $X = 1$ means low creditworthiness
  - Solvency state at end of period $S \in \{0, 1\}$
  - $\{S = 1\} = D$ 'default', $\{S = 0\} = N$ 'survival'
- Marginal distributions of X and S:
  - $x \mapsto \mathrm{P}[X = x]$ 'rating profile'
  - $p = \mathrm{P}[D] = 1 - \mathrm{P}[N]$ 'unconditional probability of default' (PD)
- Conditional marginal distributions:
  - $x \mapsto \mathrm{P}[D \,|\, X = x]$ '**PD curve**'
  - $x \mapsto \mathrm{P}[X = x \,|\, D]$, $x \mapsto \mathrm{P}[X = x \,|\, N]$ 'conditional rating profiles'

Problems in mathematical terms

- Subscript 0 for quantities related to 2009, subscript 1 for quantities related to 2010
- **Problem I.** Observed PD curve $x \mapsto \mathrm{P}_0[D \,|\, X = x]$ not positive and not monotonous $\Rightarrow$ Fit ‘smoothed’ PD curve.
- **Problem II.** Which model components from 2009 can be assumed invariant and re-used for 2010?
- **Problem III.** How to compare performance of solution approaches?
- Solution for I: ‘Quasi moment matching’ – see [Tasche(2012)].

Invariant or not?

- Rating profile $x \mapsto \mathrm{P}[X = x]$ is not invariant (as empirically observed)
- Unconditional PD p is not invariant (as empirically observed)
- **PD curve** $x \mapsto \mathrm{P}[D \,|\, X = x]$ is **not invariant**:
  - As empirically observed
  - Follows from non-invariance of p because

$$\mathrm{P}[D \,|\, X = x] \;=\; \frac{p\,\mathrm{P}[X = x \,|\, D]}{p\,\mathrm{P}[X = x \,|\, D] + (1 - p)\,\mathrm{P}[X = x \,|\, N]} \qquad (1)$$

- Can $x \mapsto \mathrm{P}[X = x \,|\, D]$ and $x \mapsto \mathrm{P}[X = x \,|\, N]$ be invariant at the same time? Unlikely – see [Tasche(2012)].

Weaker invariance assumptions

1. Default profile $x \mapsto \mathrm{P}[X = x \,|\, D]$ is invariant but survival profile $x \mapsto \mathrm{P}[X = x \,|\, N]$ is not.
2. **Likelihood ratio** $x \mapsto \lambda(x) = \frac{\mathrm{P}[X=x \,|\, N]}{\mathrm{P}[X=x \,|\, D]}$ is invariant.
3. Discriminatory power (accuracy ratio) is invariant:

$$\mathrm{AR} = \sum_{x=2}^{k} \mathrm{P}[X = x \,|\, N]\, \mathrm{P}[X \le x - 1 \,|\, D] - \sum_{x=1}^{k-1} \mathrm{P}[X = x \,|\, N]\, \mathrm{P}[X \ge x + 1 \,|\, D] \quad \text{(2a)}$$

4. Scaled PD curve: There is a constant c_{PD} such that

$$\mathrm{P}_1[D \,|\, X = x] \;=\; c_{PD}\, \mathrm{P}_0[D \,|\, X = x], \quad x = 1, \ldots, k. \qquad \text{(2b)}$$

5. **Scaled likelihood ratio:** There is a constant $c_{\mathrm{LR}} > 0$ such that

$$\lambda_1(x) \;=\; c_{\mathrm{LR}}\, \lambda_0(x), \quad x = 1, \ldots, k. \qquad \text{(2c)}$$

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Properties of the likelihood ratio

- Neyman & Pearson lemma $\Rightarrow \lambda(X)$ is the most powerful statistic for testing 'default' against 'survival'.
- PD curve and likelihood ratio are closely related:

$$\Pr[D \,|\, X = x] \;=\; \frac{p}{p + (1-p)\,\lambda(x)}, \qquad x = 1, \ldots, k. \tag{3a}$$

- Default profile and likelihood ratio are closely related:

$$\Pr[X = x \,|\, D] \;=\; \frac{\Pr[X = x]}{p + (1-p)\,\lambda(x)}, \qquad x = 1, \ldots, k. \tag{3b}$$

- (3b) $\Rightarrow$ **Unconditional PD** p is **uniquely determined** by rating profile and likelihood ratio.

Likelihood ratio and unconditional PD

Proposition 1

Let $\pi_x > 0$, $x = 1, \ldots, k$ *be a probability distribution. Assume that* $x \mapsto \lambda(x) > 0$ *is non-constant for* $x = 1, \ldots, k$*. Then*

$$\sum_{x=1}^{k} \frac{\pi_x}{p + (1-p)\,\lambda(x)} \;=\; 1 \tag{4a}$$

has a unique solution $0 \le p < 1$ *if and only it holds that*

$$\sum_{x=1}^{k} \frac{\pi_x}{\lambda(x)} \;\ge\; 1 \quad \textit{and} \quad \sum_{x=1}^{k} \pi_x\,\lambda(x) \;>\; 1. \tag{4b}$$

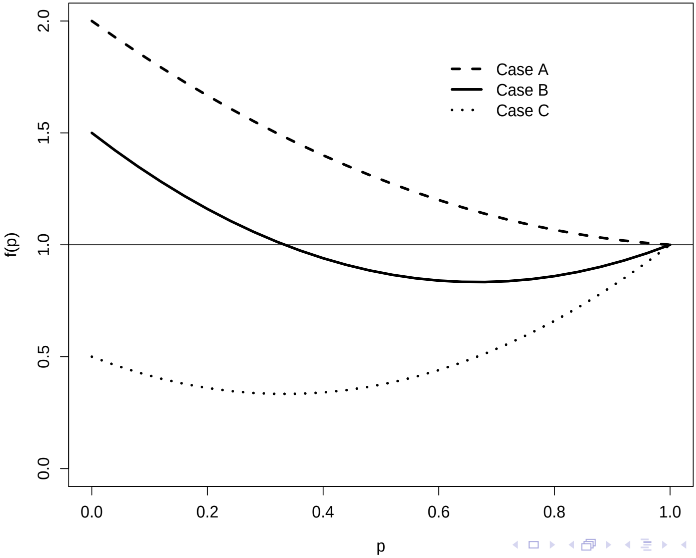
Proof. $f(p) = \sum_{x=1}^{k} \frac{\pi_x}{p+(1-p)\,\lambda(x)}$. Three cases:

A: $\sum \pi / \lambda \;>\; 1$ and $\sum \pi\,\lambda \;\le\; 1$,

B: $\sum \pi / \lambda \;\ge\; 1$ and $\sum \pi\,\lambda \;>\; 1$,

C: $\sum \pi / \lambda \;<\; 1$ and $\sum \pi\,\lambda \;>\; 1$.

Illustration of the proof of Proposition 1



Consequences of Proposition 1

- Solve (4a) with data from slide 4:
  - π_x = rating profile of 2010
  - $\lambda(x)$ = (smoothed) likelihood ratio observed in 2009
- Resulting **forecast of 2010 unconditional default rate**: 5.38%.
- Observed 2010 unconditional default rate: 1.14%.
  $\Rightarrow$ Likelihood ratio cannot be assumed to be invariant.
- Improved approach:
  - Independent estimate of 2010 unconditional default rate
  - Scaled 2009 likelihood ratio

Scaled likelihood ratio

- **Conclusion from Proposition 1:** Let $\pi_x > 0$, $x = 1, \ldots, k$ be a probability distribution. Assume that $x \mapsto \lambda(x) > 0$ is non-constant for $x = 1, \ldots, k$. Let $p \in (0, 1)$ be fixed.
  Then there is a unique number c_{LR} with

$$\left(\sum_{x=1}^{k} \pi_x \lambda(x)\right)^{-1} < c_{LR} < \sum_{x=1}^{k} \frac{\pi_x}{\lambda(x)} \tag{5a}$$

  such that

$$1 = \sum_{x=1}^{k} \frac{\pi_x}{p+(1-p)\, c_{LR}\, \lambda(x)}. \tag{5b}$$

- Solve (5b) with data from slide 4:
  - π_x = rating profile of 2010
  - $\lambda(x)$ = (smoothed) likelihood ratio observed in 2009
  - p = forecast of 2010 unconditional default rate
- Compare 2010 grade-level default rates with forecasts based on approaches 4 and 5 of slide 11.

Grade-level default rate forecasts for 2010

Observed 2010 default rates (%) vs. 'scaled PDs' and 'scaled LR' forecasts (%).

Rating	Obs. DR	Sc. PD	Sc. LR	Rating	Obs. DR	Sc. PD	Sc. LR
AAA	0	0.0007	0.0005	BBB-	0	0.2107	0.1581
AA+	0	0.0015	0.0011	BB+	0.7874	0.3006	0.2263
AA	0	0.0031	0.0023	BB	0.3623	0.4012	0.3029
AA-	0	0.0066	0.0049	BB-	0.5277	0.6024	0.4576
A+	0	0.0125	0.0093	B+	0	1.0417	0.8023
A	0	0.0241	0.018	B	0.6881	2.1134	1.6844
A-	0	0.0458	0.0342	B-	2.069	5.1671	4.5716
BBB+	0	0.0789	0.059	CCC-C	22.2727	12.7755	15.576
BBB	0	0.1307	0.0979	All	1.141	1.141	1.141

χ^2-tests of implied default profiles against observed default numbers:

- Scaled PDs: p-value 4.1%
- Scaled LR: p-value 10.5%

Outline

Observations and conclusions

- Empirical grade-level default rates:
  - Strong variation over time
  - Often zero
  - Not monotonous with regard to creditworthiness
- Therefore, making positive and monotonous default rate forecasts is challenging.
- Compared ‘scaled PD curve’ and ‘scaled likelihood ratio’ approaches:
  - Scaled LR always gives a valid PD curve.
  - Scaled LR is not ‘contaminated’ by unconditional default rate of previous year (eq. (3a)).
  - Scaled LR gives better fit of observed default rates.

Outline

S&P.
Default, Transition, and Recovery: 2009 Annual Global Corporate Default Study And Rating Transitions.
Report, Standard & Poor's, March 2010.

S&P.
Default, Transition, and Recovery: 2010 Annual Global Corporate Default Study And Rating Transitions.
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Working paper, Financial Services Authority, UK, 2012.