Monitoring Creditworthiness: Markov Chains and Replication Portfolios

Credit Scoring and Credit Control XII

Edinburgh 24-26 August 2011

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Objective of the presentation

In the previous last two Conferences (X-XI) we described a model for assessing the borrowers creditworthiness in retail portfolios (loans and credit cards)

Now some progresses will be shown:

- **Hidden Markov Models for a dynamical monitoring of portfolios taking into account the state of the credit market and the implications on credit policy**

- **Markov Chains for a dynamical monitoring of borrowers creditworthiness**

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Agenda

- INTRODUCTION
- HIDDEN MARKOV MODELS
- MARKOV CHAINS

Consum.it as part of MPS Group

- Monte dei Paschi di Siena was founded in 1472 and is the oldest Bank in the world. MPS is the 3rd Banking Group in Italy with almost 3000 branches

- Consum.it is a subsidiary owned by MPS group, being its consumer finance company

- Consum.it was founded in 1998 and now it offers a complete range of consumer finance products: its business started with purpose loans (1999), revolving credit cards (2002), personal loans (2004) and wage guaranteed loans (2009)

- Our statistical models are based on the Creditworthiness Index (CWI)

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CWI: the definition (fixed term loan)

$$X_t = \frac{\sum_{h=1}^{t} R_h (1+i)^{-h}}{\sum_{h=1}^{t} r_h (1+i)^{-h}}$$

R_h random installment at time h

$h = 1, \ldots, n$ with n term of the operation

t evaluation time with $t = 1, \ldots, n$

r_h contractual installment at time h

i internal rate of return

Quirini L., Vannucci L., (2010), "A new index of creditworthiness for retail products", in *Journal of the Operational Research Society*, **61**, 455-461.

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Some interpretations of CWI can be given to describe the customers behaviour

CWI: three interpretations

Case 1 — **Deterministic percentage of each installment**

$$R_h = a r_h \quad (a \le 1)$$

$$X_t = \frac{\sum_{h=1}^{t} R_h (1+i)^{-h}}{\sum_{h=1}^{t} r_h (1+i)^{-h}} = \ldots = a$$

Case 2 — **Systematic delayed repayment**

j = number of delayed months

$$X_{n+j} = \frac{\sum_{h=1}^{n} r_h (1+i)^{-(h+j)}}{\sum_{h=1}^{n} r_h (1+i)^{-h}} = (1+i)^{-j}$$

Case 3 — **Lottery**

p = probability of payment each installment

$$E(X_t) = E\left(\frac{\sum_{h=1}^{t} R_h (1+i)^{-h}}{\sum_{h=1}^{t} r_h (1+i)^{-h}}\right) = \ldots = p$$

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CWI is directly linked to profitability

Internal rate of return (Irr) vs CWI

$$b = \sum_{h=1}^{n} r_h (1 + x)^{-h}$$

Loan b = 12000 euro, n = 48 months, APR = 10.0%

b granted amount

n term of the operation

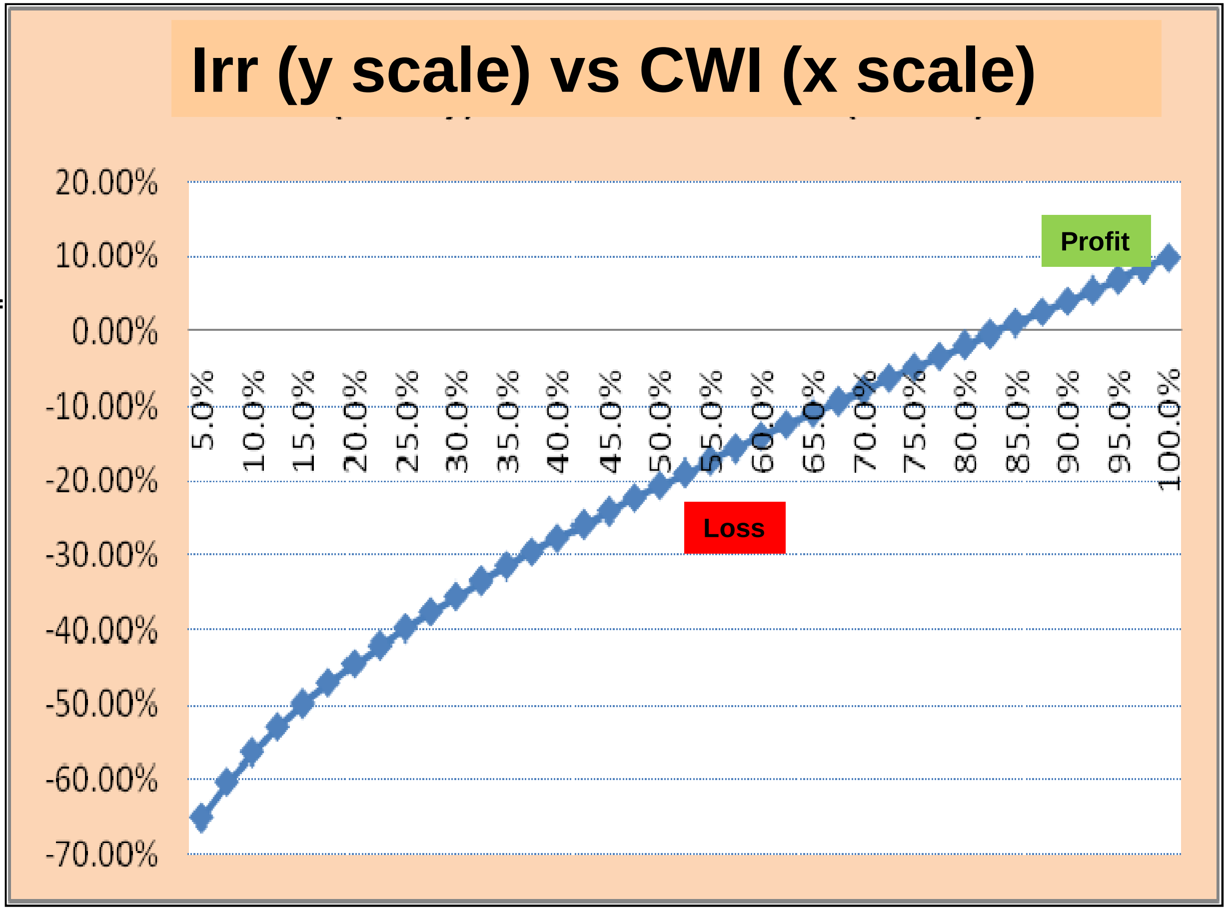
r_h installment due at time ***h***

x is the contractual rate of return

$$b = \sum_{h=1}^{n} a \cdot r_h (1 + y)^{-h}$$

a CWI (0≤ *a* ≤1)- first CWI interpretation

y is the internal rate of return



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Hidden Markov Models

Hidden Markov models (HMMs) are ones in which the underlying and hidden process is Markovian

The hidden states are linked to credit market conditions

The information at disposal are time series of the CWI

HMMs might be useful when the observations look like sampled from more than one distribution

The aim is to detect in advance the changes in the credit market conditions to modify, for instance, the cut-off levels

Data set used for the model

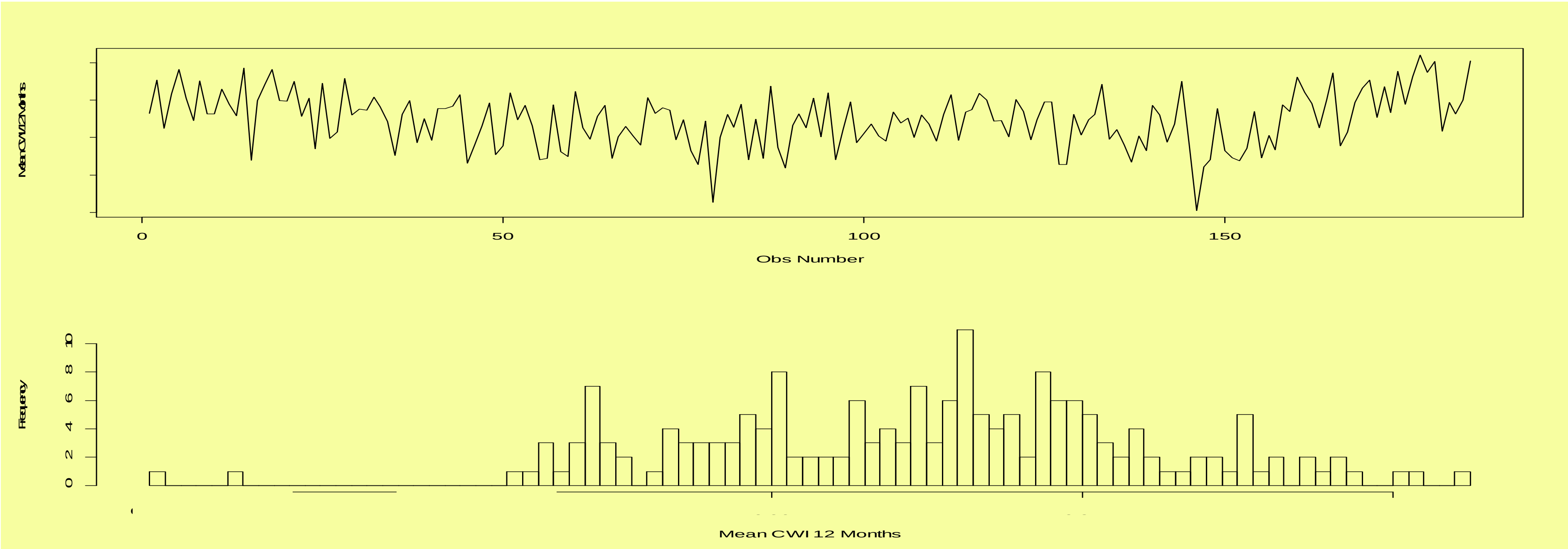
The data set is composed by personal loans granted from October 2008 until September 2009: the first semester 2009 was at high risk for the whole Italian banking system

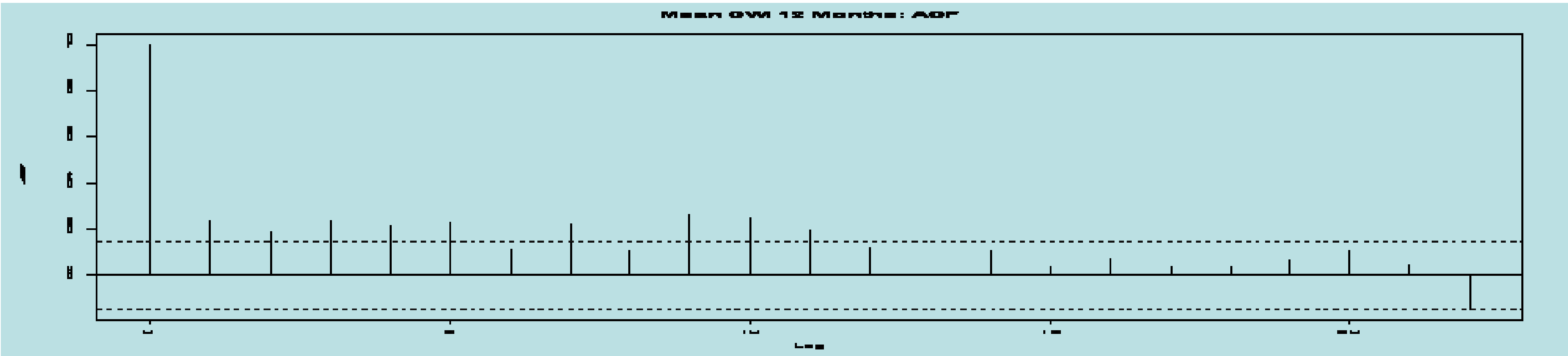
The time series has 185 elements

Each element is the mean of the CWI for 500 operations: this figure corresponds to the daily production

The histogram of data and the AutoCorrelation Function suggest to use HMMs

CWI after 12 months: time series, histogram and ACF





The standard procedure to estimate an HMM has been applied

Basic steps

The time series can be described by a gaussian HMM (or beta HMM…) and the model can be estimated by means of Baum Welch algorithm

Once the model has been estimated, the most probable global sequence of hidden states can be found by means of Viterbi algorithm

In the credit context the hidden states may be related to different credit market conditions

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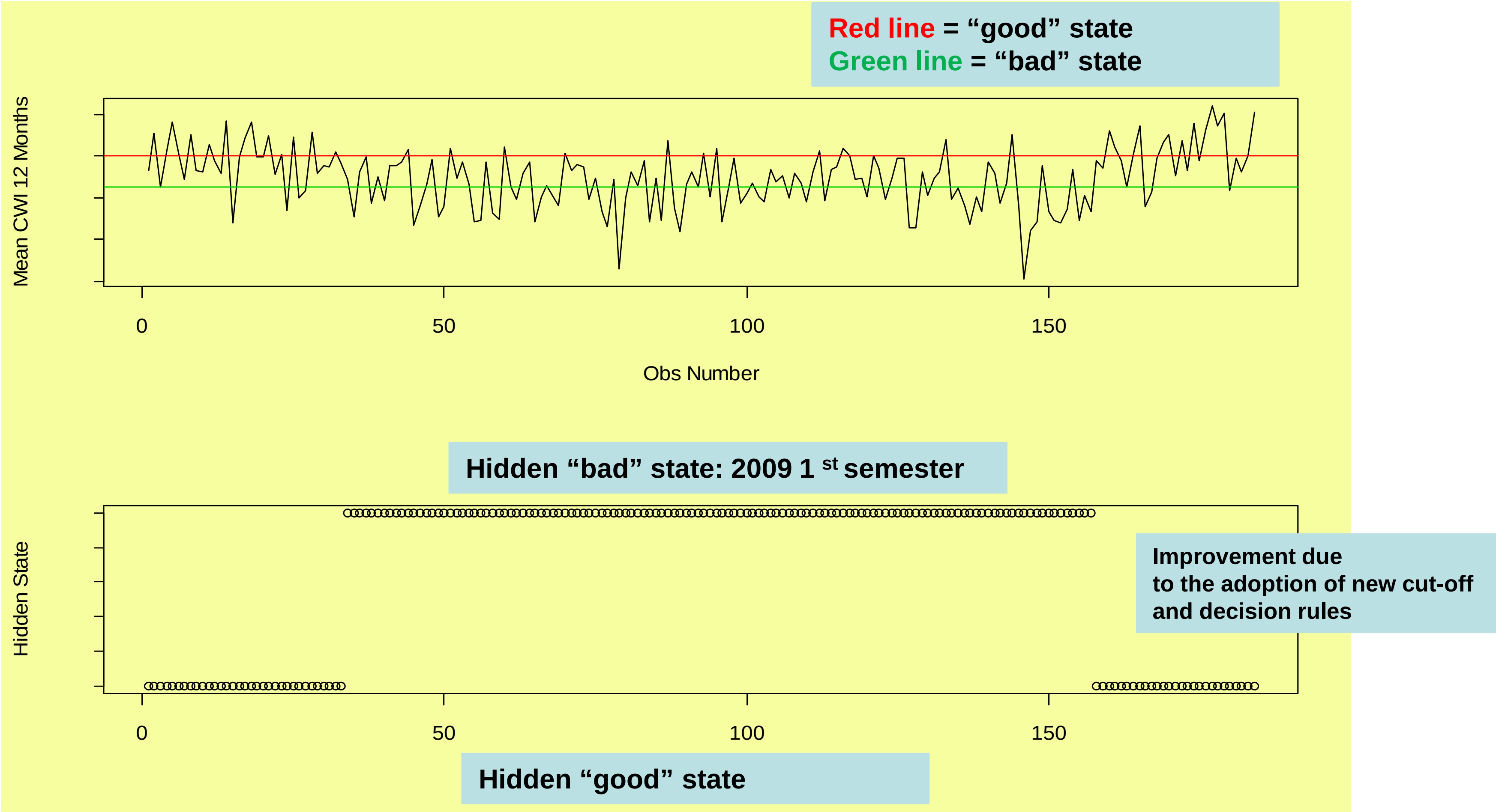
A gaussian two states HMM is fitted: the sequence of the hidden states shows high persistence

The HMM with two hidden states ("good" and "bad")



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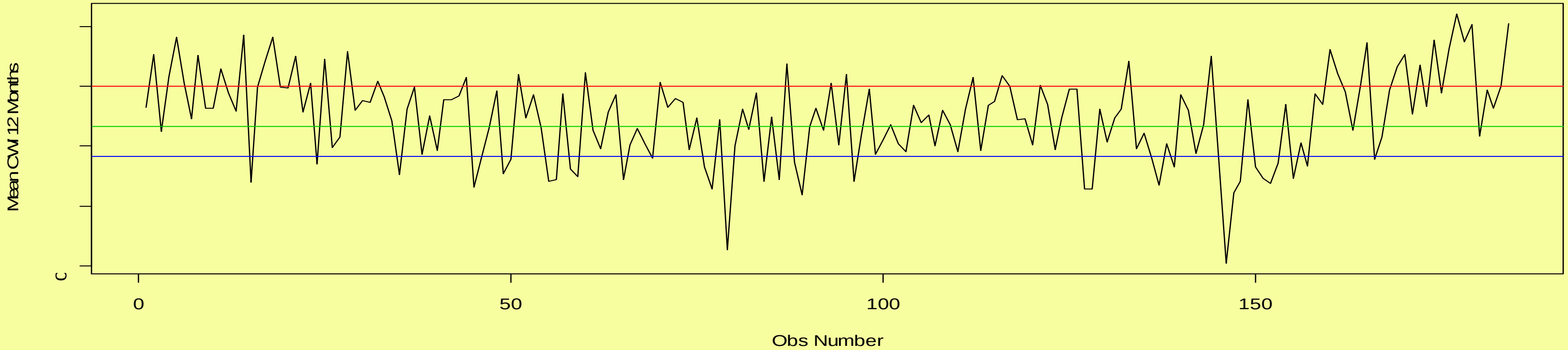
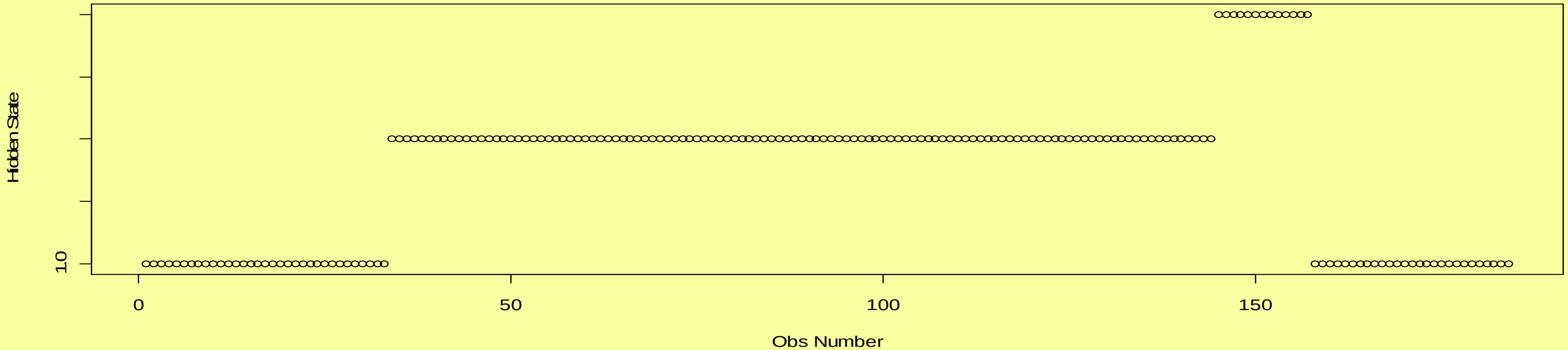
A gaussian three states HMM is also fitted: “bad” splits into a “medium” and “dangerous”

The HMM with three hidden states

Red line = “good” state
Green line = “medium” state
Blu line = “dangerous” state





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Markov chains and CWI

Markov Chains might be used to describe the evolution over a length of time of a single transaction, or for an homogeneous group of customers, with the set of states given by the number of installments in arrears

Two models have been defined, the former without an absorbing state and the latter with an absorbing one (default)

In the Conference paper it has been shown how to evaluate by means of recusive formulas the first four moments of the CWI

A simple model will be described

The model without default is defined by a transition matrix of increasing order

Parameters

$r_1 = r_2 = ... = r_n = r$ $\quad R_h = jr \quad j = 0, 1, ..., h \quad h = 1, 2, ..., n$

$S_h = \{0, 1, ..., h-1\} \quad h = 1, ..., n$

The states

$\pi_0 := P(S_{1^-} = 0) = 1$

The initial distribution

$$P_h = \begin{pmatrix} p_{h,0,0} & p_{h,0,1} & 0 & 0 \; ... \; 0 & 0 \\ p_{h,1,0} & p_{h,1,1} & p_{h,1,2} & 0 \; ... \; 0 & 0 \\ & & & ... & \\ p_{h,h-1,0} & p_{h,h-1,1} & p_{h,h-1,2} & ... \; p_{h,h-1,h-1} & p_{h,h-1,h} \end{pmatrix}$$

The transition matrix at time *h* of order *h*x(*h*+1)

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The model with default (absorbing state) has a simpler transition matrix

Parameters for the default case

$$r_1 = r_2 = \ldots = r_n = r \qquad R_h = jr \qquad j = 0, 1, \ldots, h_*$$

h_* is absorbing (here $h_*=3$)

$$S_h = \{0, 1, \ldots, h_*\} \quad \pi_0 := P\left(S_{1^-} = 0\right) = 1$$

The states and the initial distribution

$$P_1 = \begin{pmatrix} p_{1,0,0} & p_{1,0,1} \end{pmatrix} \quad P_2 = \begin{pmatrix} p_{2,0,0} & p_{2,0,1} & 0 \\ p_{2,1,0} & p_{2,1,1} & p_{2,1,2} \end{pmatrix}$$

First 2 transition matrices

$$P_h = \begin{pmatrix} p_{h,0,0} & p_{h,0,1} & 0 & 0 \\ p_{h,1,0} & p_{h,1,1} & p_{h,1,2} & 0 \\ p_{h,2,0} & p_{h,2,1} & p_{h,2,2} & p_{h,2,3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $h \geq h_*$ (here $h_*=3$)

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The CWI first four moments, at a given time *h*, depend on the the sequence of the present value of the random repayments (W_h)

CWI first four moments

$$W_h = R_1 v + R_2 v^2 + ... + R_h v^h \qquad X_h = \frac{W_h}{r_1 v + r_2 v^2 + ... + r_h v^h}$$

$$E[X_h] = \frac{E[W_h]}{r_1 v + r_2 v^2 + ... + r_h v^h} \qquad \sigma^2[X_h] = \frac{\sigma^2[W_h]}{(r_1 v + r_2 v^2 + ... + r_h v^h)^2}$$

$$\gamma_h = \frac{E[(X_h - E[X_h])^3]}{(\sigma[X_h])^3} = \frac{E[(W_h - E[W_h])^3]}{(\sigma[W_h])^3} \qquad k_h = \frac{E[(X_h - E[X_h])^4]}{(\sigma[X_h])^4} = \frac{E[(W_h - E[W_h])^4]}{(\sigma[W_h])^4}$$

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The basic quantities

$$E\left[\left(W_h - W_{k+1}\right)^m \mid S_{k+2} = j\right]$$

$$m = 1, 2, 3, 4$$

$$k = h-1, h-2, \ldots, 1, 0$$

Further details may be found on the Conference paper

A one parameter example: the model

$p_{1,0,0} = a \qquad p_{1,0,1} = 1 - a$

$$p_{h,i,j} = \begin{cases} a & \text{if } j = i \\ \frac{1-a}{2} & \text{if } j = i-1 \text{ or } j = i+1 \\ 0 & \text{otherwise } j \end{cases}$$

$$\pi_0 = (1)$$

$$\pi_1 = (a, 1-a)$$

$$\pi_2 = (a, 1-a)\begin{pmatrix} a & 1-a & 0 \\ \frac{1-a}{2} & a & \frac{1-a}{2} \end{pmatrix} = \left(a^2 + \frac{(1-a)^2}{2}, 2a(1-a), \frac{(1-a)^2}{2}\right)$$

…

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A one parameter example: the marginal distributions

h	0	1	2	3	4	5	6	7	8	9	10	11	12
0	10000												
1	9000	1000											
2	8150	1800	50										
3	7425	2438	135	3									
4	6804	2943	244	9	0								
5	6271	3341	367	20	1	0							
6	5811	3653	498	37	2	0	0						
7	5413	3893	633	58	3	0	0	0					
8	5066	4077	767	84	6	0	0	0	0				
9	4763	4214	898	114	9	1	0	0	0	0			
10	4498	4314	1025	148	14	1	0	0	0	0	0		
11	4264	4384	1146	185	20	2	0	0	0	0	0	0	
12	4056	4429	1259	225	28	2	0	0	0	0	0	0	0

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Probability x 10000

A one parameter example: the CWI first four moments

h	$E[W_h]$	$E[X_h]$	$\sigma[W_h]$	$\sigma[X_h]$	γ_h	k_h
1	0.8571	0.9000	0.2857	0.3000	-2.6667	8.1111
2	1.6825	0.9049	0.3764	0.2024	-1.8062	4.9312
3	2.4760	0.9092	0.4318	0.1586	-1.4368	4.0619
4	3.2376	0.9130	0.4694	0.1324	-1.2332	3.7556
5	3.9678	0.9165	0.4963	0.1146	-1.1104	3.6571
6	4.6672	0.9195	0.5162	0.1017	-1.0348	3.6597
7	5.3366	0.9223	0.5312	0.0918	-0.9877	3.6921
8	5.9768	0.9247	0.5428	0.0840	-0.9595	3.7472
9	6.5888	0.9270	0.5518	0.0776	-0.9470	3.8340
10	7.1734	0.9290	0.5589	0.0724	-0.9421	3.9127
11	7.7318	0.9308	0.5646	0.0680	-0.9433	3.9992
12	8.2649	0.9325	0.5690	0.0642	-0.9482	4.0338

model parameter: a = 0.9, APR = 0.05

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A one parameter example: the sensitivity analysis

a	$E[W_{12}]$	$E[X_{12}]$	$\sigma[W_{12}]$	$\sigma[X_{12}]$	γ_{12}	k_{12}
0.82	7.9877	0.9012	0.6904	0.0779	-0.9567	4.1128
0.84	8.0489	0.9081	0.6623	0.0747	-0.9501	4.0960
0.86	8.1147	0.9155	0.6330	0.0714	-0.9452	4.1113
0.88	8.1862	0.9236	0.6022	0.0679	-0.9419	4.0697
0.90	8.2649	0.9325	0.5690	0.0642	-0.9488	4.0243
0.92	8.3527	0.9424	0.5318	0.0600	-0.9802	4.0156
0.94	8.4522	0.9536	0.4872	0.0550	-1.0714	4.1078
0.96	8.5670	0.9666	0.4270	0.0482	-1.2905	4.2893
0.98	8.7019	0.9818	0.3297	0.0372	-1.9206	6.2792

$h = 12$, APR = 0.05

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Key references

Cyert, R., J. Davidson and G. Thompson (1962), Estimation of the Allowance for Doubtful Accounts by Markov Chains, *Management Science*, 8, 3-19

Thomas L., D.B. Edelman and J.N. Crook (2002), *Credit Scoring and its Applications*, SIAM Monographs on Mathematical Modeling and Computation, Philadelphia, USA

Quirini L. and L. Vannucci (2010), A New Index of Creditworthiness for Retail Credit Products, *Journal of Operation Research Society*, 61, 455-461

Zucchini W. and I.L. MacDonald (2009), *Hidden Markov Models for Time Series*, CRC Press

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