Quantum Mechanics Framework to Minimize the Lack of Stationary Properties in Markov like Credit Risk Models

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Abstract:
The recent crises in financial markets demonstrated that the lack of stationary economic properties in stochastic processes can be a limitation when it propagates into credit risk modeling. To incorporate uncertainty in stochastic metrics, we take advantage of a quantum mechanics framework to redefine the existing credit risk models in order to forecast a final risk distribution. In this work, we define the obligor as a wave function with intrinsically uncertain properties that will produce different risk measures with different probabilities. These probabilities can be derived in order to obtain a complete probability density function that incorporates the lack of stationarity of the variables. This can be achieved without the need to significantly adjust the information infrastructure that underpins the data and models.

I. Introduction

One of the greatest challenges facing the credit risk modeller is to quantify the underlying risk of the portfolio. A portfolio is a finite space; both in volume and value, and its homogeneity can always be questioned. Consequently, any statistical inference we can do over the information that is held in the portfolio will always have a great degree of error associated with it.
In recent years, Markov like models [1], [2], [3], [15] have been widely used across the industry. Essentially, a Markov type model ignores the idiosyncrasy of a single exposure, or obligor, and focus on the actual distribution of the portfolio and on the transition probabilities that have been observed – this is done with the limitations of a finite portfolio. The core assumption of the Markov approach implies to smash the variable time – we assume that all the events of classification and transition occur on the same instant, by amalgamating several periods together. By doing this we get cardinality (1000 events in 5 years become 5000 timeless events)
From a conceptual perspective, the development sample encompasses a full economic cycle – i.e., all possible states have been experienced with a reasonable probability of occurrence – ignoring the variable time would not be a major problem. Indeed, it would have some advantages, by introducing some stationarity in the model. However, the economic crisis we are currently going through has demonstrated that the vast majority of the banks has not achieved a position where detailed data for all the portfolios covering at least one economic cycle is held. Moreover, due to changes in technology, products and procedures, it is unlikely that the majority of the banks will be able to hold such data in the near future. There are obvious exceptions, and they will have to focus on getting the best out of that precious data.
Ignoring the variable time in Markov models may have a non negligible impact on the results and may result on a biased probability distribution. Additionally, associated with the uncertainty of the transition matrices due to the lack of information, there is the uncertainty of the scoring.
No matter how complex a scoring system is, there will always remain three pertinent questions.
The first one relates to the fact that a scoring system is, typically, a transformation of a multi dimensional space with characteristics that are independent of the exposure being scored, into a discrete and finite space of classes, with a grading hierarchy. This translates into the fact that, with a certain probability, exposures have been misallocated. In reality, the exposure has a probability of being mapped into each PD grade on the discrete space of classes on the scoring system.
The second one relates to the cardinality and homogeneity of the sample. The complexity of statistical approaches in recent years has allowed data limitations to be overcome, but the fact is that

the model will only be as good as the data. Additionally, the sample period also plays a very important role: exposures with different asset correlations will behave very differently, depending where we are in the economic cycle.
The third, and perhaps the less intuitive of all, is that each class must reflect a probability density function of its own[1]. The grade nature of risk classes is quite obvious, but not its analytical nature which is, at the end, what leads us to full risk quantification by a loss distribution.
To find an answer to those questions, we will change the perspective from which we look at the problem. Instead of using a scoring system to map exposures into different risk classes or segments, we will focus on the economic circumstances that surround and interact with the portfolio. We will understand how the classification of exposures depends of the economic environment and can be described in the space generated by functions that can be used to quantify risk. First we will assume the existence of functions that describe the stationarity solution of the problem – they will depend on the time variable, but that dependence is invariant in time. The next step will be to calculate a probability of assigning an exposure to the classes defined by those functions. Finally, we model the dependence of the probability values with the economic environment as if the debtor makes a trajectory in the space of the stationary solutions.

In Part II we will define the Hilbert space that is generated by the functions that were mapped into the classes, the states and the way how any exposure can be transformed into an object of that state.
In Part III we will explain how we can look at the economic conjuncture, and how the various exposures on that "bath" occupy all the states. Understandably, during an economic crisis, the higher risk states will have a higher probability of being occupied than in a soaring economy – as a consequence, if we randomly chose an obligor from the portfolio, the probability of being in a given state will depend on those factors. Essentially, this is where a century's worth of research in quantum physics becomes useful – a quantum mechanics framework will be used to formulate our problem.

II. Obligor as a quantum object

According to the Merton-Vasicek[4], [5] model, one obligor is in default when, at a fixed assessment horizon, the value of the assets is lower than its liabilities – this assumes the existence of a measure between the assets and liabilities that is specific for each obligor. It is virtually impossible to know the value of the assets and liabilities for every obligor in the portfolio and therefore we should consider this measure a theoretical concept – yet something that can be obtained.
Lets represent that difference by ϵ and let us call it the *value of the obligor.* At the heart of the economy are the players or the economic agents. If the debts of some agents will represent the assets of the others, the value of the obligor is an extensive measure, on the sense that we can have an aggregated measure of value for a group of obligors, by summing the individual value of each of the obligors in the economy. Representing the value of a portfolio with N obligors by ϵ_P, we have

$$\epsilon_P = \sum_{i=1}^{N} \epsilon_i$$

The extensibility of the measure of value associated to default, established by the Merton-Vasicek model, will be very important in the formulation of our problem. Assuming a large N and a closed portfolio, i.e., the obligors in the portfolio can only interact between themselves, the liabilities of some will be the assets of other, within the same portfolio. Under this assumption, we can state

$$\left|\frac{1}{\epsilon_i}\frac{d\epsilon_i}{dt}\right| \gg \left|\frac{1}{\epsilon_p}\frac{d\epsilon_P}{dt}\right|$$

This equation can be justified intuitively - the variation of an obligor value in the portfolio is much more volatile than the variation of the value of the entire portfolio with (large) N obligors.
In other words, on the time window we use to measure the value of the obligor, the variation of the value of the portfolio will always be much smaller and, on the same time window, we can assume that the value of the portfolio will be constant. Likewise, if we assume that the portfolio is the entire economy, we can safely say that the variation of the obligor value is much bigger than the macroeconomic variations.
The importance of this assumption will be highlighted later in the document, when we discuss the variability of macroeconomic environment.
One obligor or exposure (we will use obligor as our terminology, but in the context of the paper obligor or exposure are synonymous) is normally seen as a point in the space of financial/structural characteristics. By taking those characteristics in consideration, we make a human or mechanical decision to classify that obligor – the risk classification is a mapping of certain financial/structural into classes. When we say that one obligor has a certain risk profile, we are classifying one loss distribution function[14]

according to the probability of default and loss in the event of default.
Without any additional conjecture, we can infer that associated to each risk class there a function φ_j which will provide enough information to establish a loss distribution that characterises that class.
Naturally, that function is a result of the financial/structural characteristics of obligor α, $\varphi_k(\alpha)$, a function $\mathbb{R}^n \longrightarrow \mathbb{C}$, and we can establish that

$$\int \varphi_k{}^*(\alpha)\varphi_k(\alpha)d^n\alpha = 1$$

This condition is achieved because we are saying that an obligor is no longer a point in the space of the financial/structural characteristics, but a distribution around that point. For illustration, consider a Gaussian distribution around an average point.
To facilitate, we will use Dirac[12] notation and consider these functions as a complete basis of a vector space – a Hilbert space. By doing so, we can represent $\varphi_k(\alpha)$ as $|\varphi_k\rangle$.
The hypothesis that the model represents a complete basis is essential for what we want to demonstrate. Let us assume that each obligor can be classified on that space of functions, i.e., there is a risk classification for each individual in our portfolio. Additionally, let us also assume that it is impossible to project those functions on each other, i.e., they are orthogonal[8]. This way, we can say that we have a complete basis and that $\{\varphi_1, \varphi_2 \dots \varphi_n\}$ represents a complete orthonormal basis, that creates a functional space.
The formula

$$\int \varphi_k{}^*(\alpha)\varphi_j(\alpha)d^n\alpha = \delta_{kj}$$

will be considered a scalar product on that space and, consequently, will inherit the properties of a vector basis of any other scalar product on a complex vector space.
In this context, we can look at an obligor and, instead of mapping it to a risk class, we can describe it as a vector on this vector space and express it as a function of the basis vector, i.e.,

$$|\psi\rangle = \sum_k c_k|\varphi_k\rangle$$

The underlying assumption is that, given the obligors are not equal even if they are allocated to the same risk class, we can position them on the space of the risk classes, using the coordinates of that space.
So far, we have not made any progress from a typical scoring system. We have simply re-defined a functional object – the obligor – as a mathematical object and we described some of the maths to support the theory.
Each risk class can be seen as having a “typical” obligor, given that the function that is given to the obligor is of the same nature as the function of the obligor.
On the formula above, if only one of the coeficients c_k is different from zero, the obligor function equals the function associated with the class. We can also conclude that $0 \leq c_k \leq 1$.

Leaving aside the quantum formulation – there are several text books that can be consulted [7],[8],[9] – the scalar product can be represented as

$$\langle\varphi_k|\varphi_j\rangle = \int \varphi_k{}^*(\alpha)\varphi_j(\alpha)d^n\alpha = \delta_{kj}$$

and the vector norm on a Hilbert space can be written as

$$\langle\psi|\psi\rangle = \sum_k c_k{}^* c_k\langle\varphi_k|\varphi_k\rangle = 1$$

where $c_k{}^* c_k = |c_k|^2$ represents the probability of having the risk associated to the obligor $|\psi\rangle$ equal to the risk associated with the class $|\varphi_k\rangle$.
We designate by state vector or wave function the vector $|\psi\rangle$ that represents the obligor. In reality, according to the quantum mechanics principles that we are adopting, the function represents a state of a system that we have defined as the obligor. If the same obligor is on a different state, under different financial/structural conditions, the classification system will give a different classification from the one originally obtained.
One important aspect in the quantum formalism is the concept of observable.
The mathematical objects that are associated with classes and obligors – as we demonstrated they are similar in nature – contain information about the state of the system and the obligor. Therefore, the observable measures of the system are represented by linear operators which, when applied to a wave function on a Hilbert space, will return the values associated with the observable eigenvalues.
Previously, we have established that the obligor value $\epsilon_i$is an observable event, directly related to the event of default. In addition to that, we can describe one operator Hthat will be applied on each of the basis vector, and will return the obligor value ϵ_k associated with each of those vectors,

$$\mathrm{H}|\varphi_k\rangle = \epsilon_k|\varphi_k\rangle$$

From here, we can understand that when the operator $H$is applied to the generic obligor on a state $|\psi\rangle$ will return the values

$$\mathrm{H}|\psi\rangle = \sum_{\mathrm{k}} c_k \epsilon_k |\varphi_k\rangle$$

and, consequently, defined the expected value of H as

$$\langle \mathrm{H} \rangle = \langle \psi | \mathrm{H} | \psi \rangle = \sum_{\mathrm{k}} |c_k|^2 \epsilon_k$$

i.e., the obligor values associated with each class, weighted by the probability of having the obligor within that class. The variance will be given by $Var(H) = \langle \mathrm{H}^2 \rangle - \langle \mathrm{H} \rangle^2$.

The fact that we do not have a defined operator is not a limitation. In physics, there aren't many systems where operators that are analytically known can be used. However, one operator is represented on the space we are working on by a matrix that can be obtain via numerical or experimental methods. For now, let us assume that the operator exists and can be obtain experimentally.

In a nutshell, we do not have an obligor as a point in the space of characteristics anymore. We have a distribution around that point, and the distribution contains information about the risk of the obligor. The distribution can be derived as linear combination of functions associated with the risk classes, each of which can be regarded as template obligators

This way of seeing the problem is similar to a vector problem in the space of the obligors' functions and observables that we can obtain. The obligors are represented by linear operators on a Hilbert space generated by the functions associated with the risk classes. In practice, we have the same classification problem, but we are approaching it in a different way.

However, what we have described until now is an obligor on an isolated system, where the state vector is seen as stationary. What we want to achieve is an open system, where the macroeconomic environment can influence the model. We will have to build an obligor as a quantum open system, without disturbing what we want to keep constant, i.e., the risk classes. This can be seen as the search for a trajectory that the obligor follows on the space of the risk classes, driven by the economic environment.

III. Dynamics of risk classes space

As mentioned by Feynman[12], when we solve a problem of quantum mechanics, what we are effectively doing is dividing the world in two parts – one is the system we are interested on, the other is the rest of the universe – and consequently, we say that the group is closed. In our problem, the obligor is the system and the universe is the macroeconomic environment.

Let us assume that $\alpha$are the coordinates of the obligor in the space of characteristics and $\beta$are the coordinates of the macroeconomic environment where the obligor lies.

Let us also assume that $\{\varphi_1(\alpha), \varphi_2(\alpha) \dots \varphi_n(\alpha)\}$ are the wave functions that represent the risk classes in our problem.

As we have seen previously, the obligor as a closed system can be defined as the linear combination of the basis vectors. Because we want to keep the basis constant – the obligor will always need a classification – this means that the coefficients c_k will depend upon the macro-economic conjuncture. It is the shape of that dependency that we will need to study and obtain.

We already know that

$$|\psi(\alpha,\beta)\rangle = \sum_{\mathrm{i}} \mathrm{c_k}(\beta) |\varphi_i(\alpha)\rangle$$

If we define $|\theta_j\rangle$ as a complete basis for the system that represents the rest of the universe, the wave function (or state) of the obligor given by the economic environment is

$$|\psi\rangle = \sum_{\mathrm{ij}} \mathrm{c_{ij}} |\varphi_i\rangle |\theta_\mathrm{j}\rangle$$

If we try to derive the risk profile associated with the state of the obligor, i.e., if we apply the H operator, we will be acting over the possible states of the obligor, and not over the states of the economy – is much harder to understand the details of the latter.

Under these circumstances, we can express the operator Has

$$\mathrm{H} = \sum_{\mathrm{i\,i'\,j}} \mathrm{H_{ii'}}\ |\varphi_i\rangle |\theta_\mathrm{j}\rangle \langle \theta_\mathrm{j}| \langle \varphi_{i'}|$$

The expected value of H for a given obligor in state $|\psi\rangle$ is

$$\begin{aligned}\langle \mathrm{H} \rangle = \langle \psi | \mathrm{H} | \psi \rangle &= \sum_{\substack{\mathrm{ii'} \\ \mathrm{jj'}}} \mathrm{C_{ij}}^* \mathrm{C_{i'j'}}\ \langle \theta_\mathrm{j} | \langle \varphi_i | H | \varphi_{i'} \rangle | \theta_{\mathrm{j}'} \rangle \\ &= \sum_{ii'j} \mathrm{C_{ij}}^* \mathrm{C_{i'j}}\, \langle \varphi_i | H | \varphi_{i'} \rangle \\ &= \sum_{ii'} \langle \varphi_i | H | \varphi_{i'} \rangle \rho_{ii'}\end{aligned}$$

The section that is dependent of the economic background was isolated, and is given by

$$\rho_{ii'} = \sum_j \mathrm{C_{ij}}^* \mathrm{C_{i'j}}$$

This expression generates what is designated as a density matrix. If we look in more detail, the density matrix represents the way how a generic obligor can occupy the diverse states $|\varphi_i\rangle$ - you will remember that those diverse states are our risk classes and we know that the universe is on state $|\theta_\mathrm{j}\rangle$.
Consequently, the density matrix will reflect the state of the economy according to our risk model. This is an important aspect of our formulation and will be of significant importance from this point onwards.
To be rigorous, the density matrix is an image of the economy captured through the lenses of our risk model, given that we are still talking about a generic obligor and the vectors generated on the space are the ones of our model.
In one instant of the economic starting point, the obligor can be described by a density matrix $\rho(0)$, and the same obligor will progress through time with a dynamics[10] expressed as

$$\rho(t) = V(t)\rho(0)$$

where $V(t)$ is defined as a dynamic mapping and the values can be known based on the variation of the macroeconomic factors.
Earlier in the paper, we assumed the hypothesis that the variation of the value of the obligor is much smaller than the variation of the portfolio. In fact, the proportions are so different that we would need to measure it on a different scale – we extended this idea by saying that we can consider the changes in the portfolio null, i.e., it will be a constant portfolio. We also defined the portfolio as being comprised by N obligors. Let us add to this that the total exposure of the portfolio is E and the portfolio is, in fact, the total surrounding economy. Additionally, we can consider n, the number of score bands in our risk model.
Since each of the obligors can be on any of the n individual states, the number of states of the economy that can be achieved compatible with a total value of V, is the combination of possible N individual states that, on aggregate, will produce that total value. This means that the number of economy states that are compatible with a given aggregate value is huge compared with the number of individual possible states.
If we consider that $\Omega(\mathrm{E})$ [11] is the number of compatible economy states and $\Omega(\epsilon_k)$ the number of individual states of an obligor with a $\epsilon_k$value, then we have

$$\Omega(\mathrm{E}) = \sum_\mathrm{k} \Omega(\epsilon_k)\, \Omega(E - \epsilon_k)$$

The interpretation is relatively straightforward. The total number of possible states in the economy is the combination between all the possible states of an obligor with an obligor value of ϵ_k, and the possible states of an economy with $N-1$obligors and value $E-\epsilon_k$. Essentially, we are following Feynman doctrine and isolating one obligor from all the others.
We can give the isolated obligor the index j. From here, we can say that the probability of having obligor $j$in state $\epsilon_k$is given by the ratio between the number of states of the economy where $j$is on state $k$and the number of states of the economy where $j$is in any compatible state, i.e.

$$P(k) = \frac{\Omega(\epsilon_k)\, \Omega(E-\epsilon_k)}{\sum_\mathrm{k} \Omega(\epsilon_k)\, \Omega(E-\epsilon_k)}$$

By definition of a scoring system, we know that the obligor is in one of the $n$states, each with a value of ϵ_k. Consequently, the number of states of an obligor that is compatible with value ϵ_k is $\Omega(\epsilon_k) = 1$ - therefore we can say that the probability of having an obligor in state k is

$$P(k) = \frac{\Omega(E-\epsilon_k)}{\sum_\mathrm{k} \Omega(E-\epsilon_k)}$$

Or, with some simple algebra

$$ln\big(P(k)\big) = -ln\left(\sum_\mathrm{k} \Omega(E-\epsilon_k)\right) + ln\big(\Omega(E-\epsilon_k)\big)$$

The first parcel is a constant, where we are suming over all the $k's$- all the possible states. The second parcel, given that $E \gg \epsilon_k$, can be manipulated to show

$$ln\big(\Omega(E-\epsilon_k)\big) \approx ln\big(\Omega(\mathrm{E})\big) - v_k\left(\frac{\partial ln\big(\Omega(\mathrm{E})\big)}{\partial\epsilon}\right)_{\epsilon=E-\epsilon_k}$$

As before, the first parcel is the total number of states in the economy that does not depend of $k$and therefore can be associated with the constant we had before to obtain the normalisation constant Z. The second parcel is the one of interest.
We have

$$ln\big(P(k)\big) = -ln(Z) - \beta\epsilon_k$$

Where

$$\beta = \left(\frac{\partial ln\,(\Omega(\mathrm{E}))}{\partial\epsilon}\right)_{\epsilon=E-\epsilon_k}.$$

In our model, this represents the variation in the economy. What is expressed is that β is the variation of the number of states in the economy as a function of the value, and this is how, from our model perspective, we look at the economic background.
Because ϵ_k are constant in the model, we can say that

$$\ln\big(P(k)\big) \propto -\beta$$

If we represent the initial relevant economic indicator by β_0, each of the elements of the dynamic mapping $V(t)$ is proportional to $e^{-(\beta-\beta_0)}$. The dynamic mapping $V(t)$ is a quantum version of the transition matrix in the Markov risk models. The difference lies on how $V(t)$ will incorporate the uncertainty with classification, and the way how the economy will influence the value of the obligor. Unlike transition matrices on the traditional Markov models, the elements of the dynamic mapping are distributions and not scalar real values, which will allow us to forecast the density matrix, simply by knowing the initial density matrix $\rho(0)$.

IV. Conclusion

As we have demonstrated, by looking at the obligor as a quantum object, we can use the uncertainty on risk measurement, scoring and macro economic forecasts in our favour.
It is possible to transform traditional migration matrices into distribution matrices, that represent a super-operator, designated by dynamic mapping. This super-operator contains several components that relate to all the possible states of the economy - within a reasonable range - and eliminates the lack of stationary properties that most of the Markov models contain.
Furthermore, the forecast as a function of the economy is now possible by the re-distribution of the portfolio by the different risk classes.

This work is part of a vast investigation led by Closer, where we try to use the quantum mechanics frameworks on risk measurement. This translates into using the analytical capacity of explaining uncertainty, with the most modern data mining and statistical techniques.

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